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Sensitivity Analysis of Nonlinear Chance-constrained Problem

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1. Introduction

The problem presented here is equivalent to a stochastic linear programming problem of the chance-constrained type. In the chance-constrained problem, the constraint coefficients are normally distributed random variables. Charnes and Cooper proposed the deterministic equivalent for the stochastic linear problem. The determination of least-cost optimal cattle feed using the associated NLP was modeled by van de Panne and Popp.

The problem concerns the mixing of a number of raw materials in such a way that a hog ration is obtained that satisfies certain specified nutritive and other requirements with minimum cost for the input costs of raw materials and the requirements for the nutrients are known, the problem can be solved in a straightforward manner by deterministic linear programming methods.

The problem that arises is that the nutritive content of raw materials varies randomly from batch to batch, so that the solution given by linear programming using expected (mean) values, for instance, does not always satisfy the requirements. That is, using the expected values from a normal distribution, there is only a 50% probability that the nutritive requirements of the ration will be satisfied.

Table 1-1 gives the data for a typical case. The percentage content of protein, calcium, and phosphorus are given for 13 constituents. The cost per ton (in hundreds of dollars) is given for each constituent. The requirement of protein, calcium, and phosphorus are given for starter pigs, growers, and finishers.

The problem is to determine the mixture with minimum cost per ton that satisfies the given requirements. The problem was solved using SENSUMT computer code and sensitivity analysis are conducted with respect to the standard deviations, cost and nutritive requirement level.

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2. Deterministic Linear Programming Model

In order to compare the merits of both the linear and nonlinear models of this problem, solutions are obtained for both. First we formulate the linear problem using mean values.

If the nutritive contents and unit costs of raw materials and the requirements for the nutrients are know, this problem can be solved in a straightforward manner by deterministic linear programming methods.

The deterministic linear programming model is as follows:

Minimize $f = \sum_{i=1}^{13} r_i x_i$

subject to

$$\sum_{i=1}^{13} \bar{s}_i x_i \geq 18$$

$$\sum_{i=1}^{13} \bar{u}_i x_i \geq 1$$

$$\sum_{i=1}^{13} \bar{v}_i x_i \geq 0.9$$

$$\sum_{i=1}^{13} x_i = 1$$

$$x_i \geq 0 \quad i = 1, 2, \cdots, 13$$

where

x_i, i = 1, 2, ..., 13, denote the fraction of the mixture that is composed of each of the constituents.

r_i, i = 1, 2, ..., 13, denote the cost per ton in hundreds of dollars of each constituent.

$\bar{s}_i$, $\bar{u}_i$, and $\bar{v}_i$, i = 1, 2, ..., 13, denote the percentage content for protein, calcium, and phosphorus, respectively.

Table (1-1a) Data for Least-Cost Hog Ration Problem

Fraction x_i i=1,2,..,13	Constituent	Cost/ton ($100) r_i	% protein mean $\bar{s}_i$	% protein var. δ^2_{si}	% calcium mean $\bar{u}_i$	% calcium var. δ^2_{ui}	% phosphorus mean $\bar{v}_i$	% phosphorus var. δ^2_{vi}
x_1	Barley	0.80	ii.6	0.4844	0.05	0.0001	0.35	0.0010
x_2	Wheat	1.10	13.7	0.3003	0.07	0.	0.37	0.0009
x_3	Corn	0.85	9.5	0.14444	0.	0.	0.10	0.0001
x_4	Soy	3.45	48.5	0.0588	0.33	0.	0.62	0.0005
x_5	Mustard	2.00	31.9	4.9863	0.	0.	0.	0.
x_6	Dried milk	2.10	51.1	0.0653	1.27	0.0040	1.03	0.0021
x_7	Fish soluble	3.00	65.5	21.0222	1.27	0.1404	1.69	0.0825
x_8	Di-cal.phos.	0.80	0.	0.	23.35	1.3631	18.21	0.2073
x_9	Limestone	0.45	0.	0.	35.84	0.5138	0.	0.
x_{10}	Molasses	0.72	0.	0.	0.81	0.0289	0.08	0.0004
x_{11}	Dehy.alfalfa	1.80	21.8	0.2970	1.79	0.0097	0.31	0.0005
x_{12}	Shrimp meal	3.00	46.9	9.2933	7.34	0.3893	1.59	0.0107
x_{13}	Mono-sodium	0.60	0.	0.	0.	0.	22.45	1.0206

The solution was obtained by SENSUMT.

The optimal solution is $x^* = (0.78718, 0, 0, 0, 0, 0.172206, 0, 0, 0.0207, 0, 0, 0, 0.0199)$.

we find that a mixture of

$x_1 \doteq 0.7871$ (barley)

$x_6 = 0.1722$ (dried milk)

$x_9 = 0.0207$ (limestone)

$x_{13} = 0.0199$ (mono-sodium phosphate)

will satisfy the minimum requirements of ration for a cost of $101.26 per ton. But this ration will satisfy the requirement only 50% of the time, because the protein, calcium, and phosphorus percentage contents used were mean values.

All three nutritive requirement constraints have assocaited positive Lagrange Multipliers, and therefore, these requirements are all binding constraints at the solution.

Table (1-1b). Minimum content Required in the Feed

Type of hog	% protein	% Calcium	% phosphorus
Starter	18	1.0	0.9
Grower	16	0.8	0.7
Finisher	14	0.7	0.6

3. Nonlinear Chance-Constrined Formulation

The chance-constrained formulation of linear problem treats the nutritive requirement constraints as follows:

$$p\left[\sum_{j=1}^{n} A_{ij}x_j \geq b_i\right] \geq a_i \tag{3-1}$$

Where

p[.] : The probability operator, some or all of the coefficients A (i = 1, 2, ..., 1; j = 1, 2, ..., n) are random variables with normal distributions.

bi : Deterministic right-hand sides of constrints.

a_i : Prescribed probabilities with which the constraints must be satisfied (i = 1, 2, ..., n)

If the A_{ij}'s in (3-1) are independent normally distributed random variables with mean $\bar{A}_{ij}$'s and variance δ^2_{ij}'s, inequality (3-1) can be shown to be equivalent to

$$\sum_{j=1}^{n} \bar{A}_{ij}x_j + \Phi(a_i)\left[\sum_{j=1}^{n} \delta^2{}_{ij}x^2\right]^{1/2} \geq b \tag{3-2}$$

Where, $\Phi(a_i)$ is the percentage point or fractile corresponding to $(1 - a_i)$, and is obtained from the inverse of the standardized normal left-tail cumulative function. For example, if the i-th probability a = 0.95 $\Phi(a_i)$ is the 0.05 fractile. A value of $a_i = 0.95$ corresponds to a value of $\Phi(a_i) = -1.645$.

Inequality (3-2) is an appropriate relationship for the problem of this section. In a more general setting, however the A_{ij}'s can be dependent multivariate normal variables, in which case (3-1) can be shown to be equivalent to

$$\sum_{j=1}^{n} \bar{A}_{ij}x_j + \Phi(a_i)\left[x'Vx\right]^{1/2} \geq b \tag{3-3}$$

where

$$V = \begin{bmatrix} \text{var}(A_{il}) & \cdots & \text{cov}(A_{il}\,A_{in}) \\ \vdots & & \vdots \\ \text{cov}(A_{in}\,A_{il}) & \cdots & \text{var}(a_{in}) \end{bmatrix} \tag{3-4}$$

Inequality (3-2) is a special case of (3-3), with the off-diagonal covariance terms in (3-4) equal to zero in (3-2). Each of the stochastic constraints (3-1) must be considered in its nonlinear form (3-2). By using the means and variances listed in Table 1-1 inequality (3-2), and assuming that the probability of satisfying each of the requirements is at least 0.95, we obtain the following nonlinear model of the least-cost hog ration problem.

Minimize $\quad f = \sum_{i=1}^{13} r_i x_i$

subject to

$$\sum_{i=1}^{13} \bar{s}_i x_i + \Phi(a_i) \left[\sum_{i=1}^{13} \delta^2{}_{si} x_i \right]^{1/2} \geq 18$$

$$\sum_{i=1}^{13} \bar{u}_i x_i + \Phi(a_i) \left[\sum_{i=1}^{13} \delta^2{}_{ui} x_i \right]^{1/2} \geq 1$$

$$\sum_{i=1}^{13} \bar{v}_i x_i + \Phi(a_i) \left[\sum_{i=1}^{13} \delta^2{}_{vi} x_i \right]^{1/2} \geq 0.9$$

$$\sum_{i=1}^{13} x_i = 1$$

$$x \geq 0, \quad i = 1, 2, \cdots, 13$$

where

$\bar{s}_i, \bar{u}_i, \bar{v}_i$	:	mean value of percentage content for protein, calcium, and phosphorus, respectively.
	:	variance of percentage content for protein, calcium, and phosphorus, respectively.
$\Phi(a_i)$	:	-1.645 when minimum nutritive requirement probability is 95%.

The solution was obtained by SENSUMT.

A constrained local minimum for this problem is located at x* = (0.261418, 0, 0, 0, 0, 0.29472, 0, 0, 0.347896, 0, 0, 0.09596) at which f(x*) = 1.042

Interpreting the optimum of the NLP we find that a hog ration mixture of

x = 0.2614 (barley)
x = 0.2497 (dried milk)
x = 0.3478 (limestone)
x = 0.0959 (mono-phosphate)

will satisfy the minimum requirements for 95% of all batches mixed. The cost per ton is $104.2, which is an increase of approximately $3 over the linear program ration. But the $3 additional cost has bought a 95% probability that the hog ration requirements will be satisfied as compared to a 50% probability.

4. Sensitivity Analysis

The sensitivity analysis is conducted with respect to the standard deviation, the normal density abscissa value, cost of raw materials and the minimum percentages of protein, calcium and phosphorus.

The statement of the problem is as follow:

Minimize $f(x, \epsilon) = \sum_{i=1}^{13} \epsilon_i x_i$

subject to

$$g_1(x, \epsilon) = \sum_{i=1}^{13} \bar{s}_i x_i + \epsilon_{10} \left[\sum_{i=1}^{13} \epsilon_i^2 x_i^2 \right]^{1/2} \geq \epsilon_{35}$$

$$g_2(x, \epsilon) = \sum_{i=1}^{13} \bar{u}_i x_i + \epsilon_{10} \left[\sum_{i=1}^{13} \epsilon_i^2 x_i^2 \right]^{1/2} \geq \epsilon_{35}$$

Table (4-1) Listing of Parameters Involved in the Formulation of the Model

Standard deviation

	% Protein	% Calcium	% Phosphorus
Barley	ϵ_1	ϵ_{11}	ϵ_{22}
Wheat	ϵ_2	ϵ_{12}	ϵ_{23}
Corn	ϵ_3	ϵ_{13}	ϵ_{24}
Soy	ϵ_4		ϵ_{25}
Mustard	ϵ_5		
Dried milk	ϵ_6	ϵ_{14}	ϵ_{26}
Fish soluble	ϵ_7	ϵ_{15}	ϵ_{27}
Di-cal. phos		ϵ_{16}	ϵ_{28}
Limeston		ϵ_{17}	
Molasses		ϵ_{18}	ϵ_{30}
Dehy.alfalfa	ϵ_8	ϵ_{19}	ϵ_{31}
Shrimp meal	ϵ_9	ϵ_{20}	ϵ_{32}
Mono-sodium			ϵ_{33}

Minimum content Required in the feed		Normal density of Abscissa value		Cost of raw Materials	
% Protein	ϵ_{35}	% Protein	ϵ_{10}	Barley	ϵ_{37}
% Calcium	ϵ_{36}	% Calcium	ϵ_{21}	Dried milk	ϵ_{38}
% Phosphorus	ϵ_{29}	% Phosphorus	ϵ_{34}	Limestone	ϵ_{39}
				Monosodium	ϵ_{40}

$$g_2(x,\ \epsilon) = \sum_{i=1}^{13} \bar{u}_i x_i + \epsilon_{10} [\sum_{i=1}^{13} \epsilon_i^2 x_i^2]^{1/2} \geq \epsilon_{35}$$

$$g_3(x,\ \epsilon) = \sum_{i=1}^{13} \bar{v}_i x_i + \epsilon_{32} [\sum_{i=1}^{13} \epsilon_i^2 x_i^2]^{1/2} \geq \epsilon_{36}$$

$$h(x,\ \epsilon) = \sum_{i=1}^{13} x_i = 1$$

$$x_i \geq 0,\ i = 1,\ 2,\ \cdots,\ 13$$

Here, the parameters in the sensitivity analysis are denoted ϵ_i. The listing and description of the parameters involved in the formulation of the model are given in the Table (4-1), the problem data for hog ration problem and the initial starting point for the SENSUMT program are shown in Table (1-1) and (4-1). As indicated in the table, the standard deviation, the normal density abscissa value, cost of raw materials and the minimum percentage of nutritients are treated as parameters in conducting the sensitivity analysis.

Table (4-2) gives the computer solution and the first partial derivative of the optimal value function $f(x^*)$ with respect to the parameter of standard deviations.

When considering the partial derivatives with respect to the standard deviation, as might be expected the input with a large standard deviation, the standard deviation of protein content in barley, has the greatest effect on solution.

Table (4-2) Otpimal Value Function Derivatives with Respect to Standard Deviations

		Barley	Dried milk	Lime-stone	Monosodium	Others
Protein	$\partial f/\partial \epsilon_i$	0.0041	0.007	0.	0.	0.
Calcium	$\partial f/\partial \epsilon_i$	0.	0.	0.	0.	0.
Phosphorus	$\partial f/\partial \epsilon_i$	0.56-06	0.15-05	0.	0.77-04	0.
Optimal mix		0.2614	0.2947	0.3479	0.0959	0.

Thus, changes in parameter 1 can affect the cost of hog ration a significant amount and consequently. The value of parameter 1, the standard deviation of the distribution of protein content in barley, should be known with a high precision. This may be achieved by additional sampling and testing of barly.

Table (4-3), (4-5) gives the optimal mixture in terms of starter, grower and finisher hog and the first partial derivatives of the optimal value function f(x*) with respect to nutritive content requirement.

Changes in nutritive requirements do not change the mixture constitutents. The mixture constituents are the same, although their mixture fractions have changed slightly. As expected, an decrease in minimum nutritive requirement causes an decreased cost of hog ration.

As a matter of interrst, the partial derivatives of the objective function with respect to minimum nutritive requirement, the parameter 35, 36 and 29, are actually the Lagrange Multipliers assocaited with the three inequality constraints.

Table (4-3) Optimal Mixture and Solution

Opt. Mix hog type	Barley	Dried-milk	Lime-stone	Mono-sodium	Others	f(x*) ($/ton)
Starter	0.2614	0.2947	0.3479	0.095	0.	$104.2
Grower	0.2804	0.2518	0.3709	0.089	0.	$97.7
Finisher	0.2937	0.2102	0.4101	0.0858	0.	$91.3

The values of the partial derivatives in the sensitivity analysis are shown on Table (4-4). The values of the Lagrange Multipliers associated with the three inequality constraints which are estimated in the SENSUMT output are, ul = 0.97503-02, u2 = 0.3220-08, u3 = 0.50469-02.

Table (4-4) Optimal Value Function Derivatives with Respect to Minimum Nutritive Requirements

Type of hog constraint	Derivatives Value			Constrains value		
	Starter	Grower	Finisher	Starter	Grower	Finisher
Protein	0.00975	0.00516	0.00273	0.39-05	0.72-05	0.14-05
Calcium	0.32-08	0.29-08	0.27-08	11.56	12.79	13.95
Phosphorus	0.00505	0.08283	0.00766	0.74-05	0.13-05	0.49-05

The values computed two different ways are in close agreement. Investigation of this mixture reveals that the calcium constraint is not binding at the solution. The actual amount of calcium in the mixture is approximately more than 10 times the amount required. If this is deemed to be too high a content for proper growth regulation in pigs, an additional constraint can be added to the problem giving an upper bound to the requirement. In fact, all requirements can be required to fall within certain upper and lower bound. The requirements can also be forced to hold certain ratios to each other within the mixture by the addition of appropriate equality constraints.

The value of Lagrange Multiplier of the requirement for protein is approximately 2 times larger than the phosphorus, indicating that the price per ton of the total ration is more sensitive to the requirement for protein than for phosphorus. But the magnitude of both multipliers show relative insensitivity to both requirements for small changes. That is, if the requirement for protein was reduced by one unit from 18 to 17, an approximate savings of $0.98 could be obtained on the price of the mixture.

Table (4-5) gives optimal mixtures in accordance with the changes in parameters 10.21 and 34. The probability that the nutritive content constraint is satisfied. As expected, an increase in the probability causes an increase cost of the hog ration.

Table (4-5) Optimal Mixture in Accordance with the Nutritive Requirement Probability

Prob.	Barley	Dried-milk	Lime-stone	Mono-sodium	Others	f(x*)
50% $\Phi = 0$	0.782	0.172	0.0207	0.0199	0.	$101.2
95% $\Phi = -1.645$	0.2914	0.2947	0.3499	0.0959	0.	$104.2

An increase in parameter 10. 21 and 34, the probability that the protein, calcium and phosphorus content is satisfied, actually makes the value of Φ more negative due to the way it is coded here.

As we know, the 50% probability case is linear programming model. The cost per ton is $104.2, which is an increase of approximately $5 over the linear programming ration. But the $3 additional cost has bought a 95% probability that the ration requirements will be satisfied as compared to a 50% probability. The mixture constituents (of the nonlinear model, 95%) are the same (as the nonlinear model, 50%), although their mixture fractions have changed drastically.

Table (4-6) Optimal Valve Function Derivatives with Respect to the Nutritive Requirement Probability

		Nutritive Requirement Probability	
		50%	95%
Protein	$\partial f/\partial \epsilon$	-0.00549	-0.00248
Calcium	$\partial f/\partial \epsilon$	-0.162-05	-0.575-09
Phosphorus	$\partial f/\partial \epsilon$	-0.0307	-0.00507

According to Table (4-6), optimal value function is sensitive to the phosphorus requirement probability.

Table (4-7) also suggests that the otpimal solution value is very sensitive to the cost of raw materials.

Table (4-7) Optimal Value Function Derivatives

		Most sensitive Partial Derivatives	Most sensitive Nutrients
Standard Deviation	$\frac{\partial f}{\partial \epsilon_1}$	0.0041	Protein in Barley
Nutritive Requirement	$\frac{\partial f}{\partial \epsilon_{35}}$	0.0098	Protein
Requirement Probability	$\frac{\partial f}{\partial \epsilon_{34}}$	0.0051	Phosphorus
Cost of Raw Material	$\frac{\partial f}{\partial \epsilon_{39}}$	0.3479	Limestone

Table (4-7) results indicate that the optimal value function is sensitive to parameters 1, 35, 34, and 39.

5. Parametric Optimal Value Bounds Analysis

Consider the following right-hand side parametric programming problem of the chance-constrained NLP model

$$\text{Minimize} \quad f(x) = \sum_{i=1}^{13} r_i x_i$$

subject to

$$\sum_{i=1}^{13} \bar{s}_i x_i - 1.645 \left[\sum_{i=1}^{13} \delta^2{}_{si} x_i{}^2 \right]^{1/2} \geq \epsilon_{35}$$

$$\sum_{i=1}^{13} \bar{u}_i x_i - 1.645 \left[\sum_{i=1}^{13} \delta^2{}_{ui} x^2 \right]^{1/2} \geq \epsilon_{36}$$

$$\sum_{i=1}^{13} \bar{v}_i x_i - 1.645 \left[\sum_{i=1}^{13} \delta^2{}_{vi} x_i \right]^{1/2} \geq \epsilon_{29}$$

$$\sum_{i=1}^{13} x = 1$$

$$x_i \geq 0, \quad i = 1, 2, \cdots, 13$$

This problem has a linear objective function. Thus, f(x) can be concave and convex and the nonlinear constraints can be shown to be concave. So the problem $R(\epsilon)$ is convex and will be designated by $CR(\epsilon)$. It is well known that $f^*(\epsilon)$, the optimal value function of the problem $CR(\epsilon)$, is convex function of ϵ.

The convexity of the optimal value function $f^*(\epsilon)$ of the problem $CR(\epsilon)$ enable us to calculate parametric upper and lower bounds on this function when any of the problem parameters is radically perturbed.

Computer solutions of the perturbed problems and bounds on the optimal value function for the right-hand side parameter, the nutritive requirement are shown on Fig. 5-1.

The graphical depiction of the bounds derived for $f^*(\epsilon)$ as a function of parameter are shown on Fig. 5-2.

We will prove that the left-hand side of the nonlinear constraint

$$g(x) = a'x + \Phi(x'Vx)^{1/2}$$

is a strictly concave function for all vectors we consider. If g(x) is strictly concave, then the following relation should be true for any two vectors x^1 and x^2, and $0 < A < 1$,

$$a'[Ax^1 + (1-A)x^2] + \Phi([Ax^1 + (1-A)x^2]'V[Ax^1 + (1-A)x^2])^{1/2} >$$

$$A(a'x^1 + \Phi[(x^1)'Vx^1]^{1/2}) + (1-A)(a'x^2 + \Phi((x^2)'Vx^2)^{1/2})$$

If we cancel the linear terms on both sides and divide by Φ, ($\Phi < 0$), we obtain

$$((Ax^1 + (1-A)x^2)' V (Ax^1 + (1-A)x^2)^{1/2} < A((x^1)' V x^1)^{1/2}$$

$$+ (1-A)((x^2)' V x^2)^{1/2})$$

After squaring, cancelling some terms and division by 2A(1-A) we have

$$(x^2)' V x^1 < ((x^1)' V x^1 (x^2)' V x^2)^{1/2}$$

If we put $V = S'S$, $s^1 = Sx^1$, and $s^2 = Sx^2$

then $\qquad ((s^2)'s^1) < (s^1)'s^1 (s^2)'s^2.$

which is the Schwarz inequality for the case $s^1 \neq cs^2$ with c any constant. It is obvious that it is possible to repeat the same sequence of relations in backward direction, so that f(x) is a strictly concave function for any two vectors $x^1 \neq cx^2$, c being again any constant. Because the linear equation in the constraints prevents the existence of any two feasible vectors x^1 and x^2, such that $x^1 = cx^2$, g(x) is strictly concave for all vectors satisfying the equality constraint.

OPTNAL VALUE FUNCTION BOUNDS WHEN PAR(29) IS PERTURBED

POINT 1 (UNPERTURBED SOLUTION):

PAR(29) = 0.9000000D+00 F(PAR(29)) = 0.1025814D+01

POINT 2 (PERTURBED SOLUTION):

PAR(29)) = 0.4000000D+01 F(PAR(29) = 0.1050830D+01

LINE UNDER ESTIMATING F* AT POINT 1

F = 0..5241818D-02 * PAR(29) + 0.1021096D+01

LINE UNDER ESTIMATING F* AT POINT 2

F = 0.8433453D-02 * PAR(29) + 0.1017096D+01

F* BOUND EVALUATION AT TEN EQUIDISTANCE POINTS BETWEEN POINTS 1 AND 2

PAR(29)	LOWER BOUND	UPPER BOUND
0.900	0.1025914D+01	
1.210	0.1027439D+C1	
1.520	0.1029915D+01	
1.830	0.1032529D+C1	
2.140	0.1035144D+C1	
2.450	0.1037758D+01	
2.760	0.1040372D+01	
2.070	0.1042987D+01	
3.070	0.1042987D+01	
3.380	0.1045601D+01	
3.690	0.1048216D+01	
4.000	0.1050830D+01	

Fig. 5-1. Parametric lower bounds on f* (ϵ_{29})

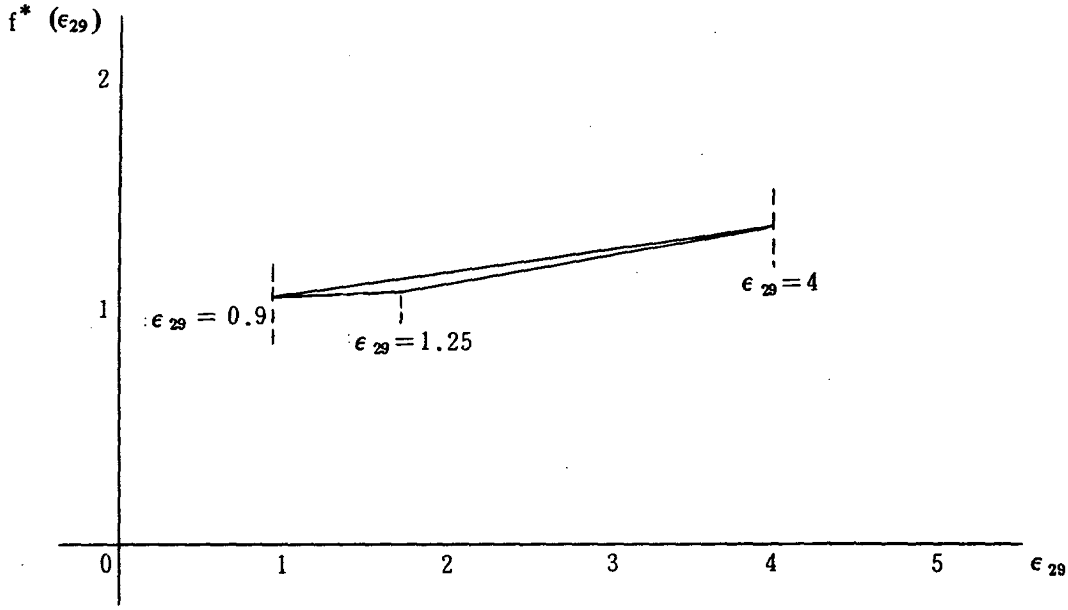

Fig. 5-2. Graph of bounds on $f^*(\epsilon_{29})$

6. Conclusions

The results of sensitivity analysis gives us that the price per ton of optimal hog ration is not sensitive to the standard deviation of nutritive content of raw materials. So we don't need to conduct more sampling to get sharper estimate.

We find out that only \$3 additional cost per ton of hog ration has bought a 95% probability that the hog ration requirements will be satisfied as compared to 50% probability.

The actual amount of calcium in the otpimal mixture of hog ration is more than 10 times the amount required. If this is deemed to be too high, an additional constraint can be added to the problem giving an upper bound to be required.

The price per ton of total ration is relative insensitive to the nutritive requirement.

It is shown that the right-hand side parametric programming of this hog ration problem is convex and derived parametric upper and lower bounds on the optimal value function for the parameters of the nutritive content requirement.

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