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Congested Market Equilibrium Analysis

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Abstract

Congestion occurs whenever users interfere with each other, while competing for scarce resources. In a congested market, such as a telecommunication service market, users of telecommunication services incur costs in using the service in addition to the price. The user's own time costs involved in learning to use the service, waiting for the service, and making use of the service are typically greater than the price of telecommunication services.

A market equilibrium analysis is performed in which a method for user demand aggregation is developed. The effects of price changes on user demands and market demands for congested services are examined. It is found that total market demands may increase as the price for less-congested services increase under certain demand conditions. This suggests that a nonuniform pricing scheme for a congested service may improve the utilization of the congested system. The sign of price cross-elasticity for congested services is show to vary with demand conditions. A possible complementary property of congested services is found and the implication of such a property is discussed. It is argued that such a complementary property may lead to a cross subsidy in a market with congestion.

Finally, comparisons between uniform pricing and nonuniform pricing policies are made. A specific numerical example is given to show that a nonuniform pricing policy may be Pareto superior to a uniform pricing policy.

1. Introduction

Congestion occurs whenever users interfere with each other, while competing for scarce resources. In a congested market, such as a telecommunication service market, users of telecommunication services incur costs in using the service in addition to the price. The user's own time costs involved in learning to use the service, waiting for the service, and making use of the service are typically greater than the price of telecommunication services.

In a market where services are subject to congestion, congestion can be thought of as creating quality differences among otherwise indistinguishable services. Congestion is an externality, because time is a cost to users, but is not revenue to providers. The problem is complicated by the fact that the quantity demanded by users typically depends on the price

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and the user time cost, which in turn depends on the level of congestion. For a given set of services and prices set by providers, the equilibrium levels of congestion are determined through user-optimizing behavior.

The theory of congested systems has primarily focused on the control of congestion through the pricing of services in nonmarket situations, such as highways or airports. An extensive literature review appears in Agnew [1]. Diamond [5] examines the problem of pricing congested facilities in the presence of consumption externalities in a general context. Eric [10], Lecaros, and Dunn [7] present an equilibrium theory for markets in which a number of competing firms offer services subject to congestion. However, these studies are limited to the case where all users have the same value of time. In most markets users will have different values of time that they will use in estimating their costs for time spent consuming the service.

In the absence of user heterogeneity with respect to value of time, equilibrium will be reached when all users incur the same user cost for the service [7, 10]. In the presence of user heterogeneity with respect to value of time, at equilibrium, it is shown that users with high values of time incur a higher user cost than users with low values of time [13].

Starting from the analysis of an individual user's optimizing behavior in congested markets, market demand functions for congested services are derived from the aggregation of individual user demands, and the structure of the derived market demand functions are characterized. The analysis performed here allows for differences in both the value of user time and demand among a user population.

Special features of the effects of price on the user demands and market demands are examined. One of the interesting features observed in congested markets is that the total demand for both a higher-priced service (less-congested) and a lower-priced service (more-congested) may increase if the price of the less-congested service increases. This suggests that a nonuniform pricing scheme for congested services may improve the utilization of the congested system.

Finally, comparisons between uniform pricing and nonuniform pricing policies are made using a numerical example.

2. User's Optimizing Behavior

Users of a congested service incur costs in using the service in addition to the price. The user's own time costs involved learning to use the service, waiting for the service, and making use of the service are typically greater than the price of such telecommunication services as telephone service or electronic message service. Other user costs can also be important but are not treated explicitly in this paper.

Total user cost per unit of service, u, can be expressed as the sum of price, p, and time cost, c, which is the product of user time value, w, in dollars per hour, and the amount of time required to consume one unit of service, τ, in hours, Thus,

$$u = p + c = p + w\tau$$

Users view u as the decision variable in choices with respect to the quantity of service purchased, rather than p. Service providers, on the other hand, can only control price. Providers, therefore, seek information about consumer behavior in the form of conventional demand curves which plot quantity purchased as a function of price.

Let u_j^k be the unit cost of service j to user k. User k has time value w^k per unit time and takes time τ_j to consume one unit of service, where τ_j is taken to be a function of the total amount of service provided by provider j but independent of the user k. Then

$$u_j^k = p_j + c_j^k = p_j + w^k \tau_j (x_j)$$

One service is said to be more congested than another, if the unit congestion costs, τ_j, of the two services have the same functional form and the more congested service has a higher unit congestion cost.

The user's decision problem will be stated under the assumption that user k has a benefit function B^k which is his monetary willingness to pay for the total quantity of service q^k that he consumes. Given a set of n services, user k maximizes net benefit, the difference between B^k and user cost. His decision problem, assuming a concave benefit function, is then

$$\max_{q_1^k, \cdots, q_n^k} B^k(q^k) - \sum_j u_j^k q_j^k \qquad (1)$$

subject to

$$q^k = \sum_j q_j^k, \quad j = 1, \cdots, n \qquad (2)$$

$$u_j^k = p_j + w^k \tau_j \qquad (2)$$

$$q_j^k \geqq 0, \quad j = 1, \cdots, n \qquad (4)$$

The optimization leads to the relationships

$$q_j^k \geqq 0 \text{ if } \frac{\partial B^k}{\partial q_j^k} = u_j^k \qquad (5)$$

$$q_j^k = 0 \text{ if } \frac{\partial B^k}{\partial q_j^k} < u_j^k \qquad (6)$$

If u^k is the equilibrium (unit) user cost, then u^k is the minimum user cost

$$u^k = \min_j u_j^k \qquad (7)$$

At such a demand equilibrium, no user has an incentive to modify his choice of services or the quantities of service purchased, and supply equals demand.

$$x_j = \sum_k q_j^k \qquad (8)$$

$$q^k = \sum_j q_j^k \qquad (9)$$

At such an equilibrium, the demand of every user group is allocated among the services in a way that yields the minimum user cost.

3. Market Demand Functions for Congested Services

In this section, market demand functions for congested services are derived from the aggregation of individual user demands, and the structure of the derived market demand functions are characterized. The derivation of the demands for congested services is based on the demand equilibrium conditions obtained from the analysis of user optimizing behavior in section 2.

Consider now a simple case in which only two services are offered, each of which is characterized by a price, p_j, and a unit congestion cost, τ_j, which is a function of the amount of service provided, x_j. Congestion costs are further assumed to be linear functions of the amount of service provided

$$\tau_j = \alpha_j x_j, \quad j = 1, 2 \qquad (10)$$

where the coefficient α_j is positive.

In order to explore some of the effects of price on demand, it will temporarily be assumed, further, that both services have the same coefficient α. This condition will be relaxed, in making a sample computation below.

The quantity of service demanded by user group k, q^k, is assumed to be linear in user cost as follows.

$$q^k = a_k - b_k u^k, \quad k = 1, 2 \qquad (11)$$

where a_k and b_k are positive constants. Thus, the users are inhomogeneous both with respect to the value of user time and their demand functions.

Four different cases can exist at demand equilibrium, depending on the prices and user time values.

Case I: Service 1 (the lower priced service) captures all the demands for services by both user group.

In general, a reduction in price may not capture all the demand in a congested market. But in this case, one service has captured the entire market demand. Since users choose services by user cost, under this case both user groups incur a least user cost for service 1. The demand for service 2 is thus zero and the demand for service 1 is the sum of the demands by both user groups. The corresponding demand equilibrium conditions are

$$u^1 = p_1 + w^1 \alpha x_1 \leqq p_2 + w^1 \alpha x_2 \qquad (12)$$

$$u^2 = p_1 + w^2 \alpha x_1 \leqq p_2 + w^2 \alpha x_2 \qquad (13)$$

$$x_1 = \sum_{k=1}^{2} (a_k - b_k u^k) \qquad (14)$$

$$x_2 = 0 \qquad (15)$$

Substituting u^1 and u^2 from Eqs. (12) and (13) into Eq. (14) and solving for x_1 yields

$$x_1 = (1 + b_1 w^1 \alpha + b_2 w^2 \alpha)^{-1} [(a_1 + a_2) - (b_1 + b_2) p_1] \quad (16)$$

which is equal to the total market demand for services. The user costs, u^1 and u^2 can be obtained as a function of p_1 by substituting x_1 from Eq. (16) into Eqs. (12) and (13), respectively:

$$u^1 = (1 + b_1 w^1 \alpha + b_2 w^2 \alpha)^{-1} [(a_1 + a_2) w^1 \alpha + (1 + b_2 \alpha (w^2 - w^1)) p_1] \quad (17)$$

$$u^2 = (1 + b_1 w^1 \alpha + b_2 w^2 \alpha)^{-1} [(a_1 + a_2) w^2 \alpha + (1 + b_1 \alpha (w^1 - w^2)) p_1] \quad (18)$$

The user demands, q^1 and q^2 can be obtained as a function of p_1 by substituting u^1 and u^2 from Eqs. (17) and (18) into Eq. (11) :

$$q^1 = (1 + b_1 w^1 \alpha + b_2 w^2 \alpha)^{-1}$$
$$\times [a_1 + a_1 b_2 w^2 \alpha - a_2 b_1 w^1 \alpha - b_1 (1 + b_2 \alpha (w^2 - w^1)) p_1] \quad (19)$$

$$q^2 = (1 + b_1 w^1 \alpha + b_2 w^2 \alpha)^{-1}$$
$$\times [a_2 + a_2 b_1 w^1 \alpha - a_1 b_2 w^2 \alpha - b_2 (1 + b_1 \alpha (w^1 - w^2)) p_1] \quad (20)$$

Define R_1 as the set of prices for which this case can occur, then we can find the set of prices R_1 which satisfies constraints (12), (13), and the nonnegativity of the user demands, q^1 and q^2 defined in Eqs. (19) and (20), respectively.

Case2: User group 1 consumes both services, and user group 2 consumes service 2 only.

This case can happen if user group 1 incurs the same user cost for both services and user group 2 incurs a least user cost for service 2. The corresponding demand equilibrium conditions are

$$u^1 = p_1 + w^1 \alpha x_1 = p_2 + w^1 \alpha x_2 \quad (21)$$

$$u^2 = p_2 + w^2 \alpha x_2 \leqq p_1 + w^2 \alpha x_1 \quad (22)$$

$$x_1 + x_2 = \sum_{k=1}^{2} (a_k - b_k u^k) \quad (23)$$

Substituting u^1 and u^2 from Eqs. (21) and (22) into Eq. (23) and using the relationship between x_1 and x_2 from Eq. (21), we obtain the market demand functions, x_1 and x_2 in terms of prices, p_1 and p_2:

$$x_1 = (2 + b_1 w^1 \alpha + b_2 w^2 \alpha)^{-1}$$
$$\times \left[(a_1 + a_2) - \frac{1 + b_1 w^1 \alpha + b_2 w^2 \alpha}{w^1 \alpha} p_1 + \frac{1 + b_2 \alpha (w^2 - w^1)}{w^1 \alpha} p_2\right] \quad (24)$$

$$x_2 = (2 + b_1 w^1 \alpha + b_2 w^2 \alpha)^{-1}$$

$$\times [(a_1 + a_2) + \frac{1}{w^1 \alpha} p_1 - \frac{1 + b_1 w^1 \alpha + b_2 w^1 \alpha}{w^1 \alpha} p_2] \quad (25)$$

The user costs, u^1 and u^2 can now be expressed as a function of p_1 and p_2 by substituting x_1 and x_2 from Eqs. (24) and (25) into Eqs. (21) and (22), respectively:

$$u^1 = (2 + b_1 w^1 \alpha + b_2 w^2 \alpha)^{-1} \times [(a_1 + a_2) w^1 \alpha + (1 + b_2 \alpha (w^2 - w^1)) p^2 + p_1] \quad (26)$$

$$u^2 = (2 + b_1 w^1 \alpha + b_2 w^2 \alpha)^{-1}$$

$$\times [(a_1 + a_2) w^2 \alpha + [\frac{w^2}{w^1}] p_1 + [2 - \frac{w^2}{w^1}) + b_1 \alpha (w^1 - w^2)] p_2] \quad (27)$$

Equivalently, the user demands, q^1 and q^2 can be expressed as functions of p_1 and p_2:

$$q^1 = (2 + b_1 w^1 \alpha + b_2 w^2 \alpha)^{-1} \times [(2 a_1 + a_1 b_2 w^2 \alpha - a_2 b_1 w^1 \alpha) - b_1 p_1 - b_1 [1 + b_2 \alpha (w^2 - w^1)] p^2] \quad (28)$$

$$q^2 = (2 + b_1 w^1 \alpha + b_2 w^2 \alpha)^{-1} [(2 a_2 + a_2 b_1 w^1 \alpha - a_1 b_2 w^2 \alpha)$$

$$- \frac{b_2 w^2}{w^1} p_1 - b_2 [b_1 \alpha (w^1 - w^2) + (2 - \frac{w^2}{w^1})) p^2] \quad (29)$$

The demand for service 2 by user group 1, q_2, can be determined by subtracting x_2 from q^2. Finally, the total market demand x can be obtained by summing x_1 and x_2 or q^1 and q^2:

$$x = (2 + b_1 w^1 \alpha + b_2 w^2 \alpha)^{-1}$$

$$\times [2 (a_1 + a_2) - \frac{b_1 w^1 \alpha + b_2 w^2 \alpha}{w^1 \alpha} p_1 + \frac{b_2 \alpha (w^2 - 2 w^1) - b_1 w^1 \alpha}{w^1 \alpha} p_2] \quad (30)$$

Define R_2 as the set of prices for which this case can occur, then we can find the set of prices R_1 which satisfied constraint (22), and the nonnegativity of x_1, x_2, q^1, q^2, and q^1_2 defined above.

The market demand for service 1 (service 2) decreases as the price of service 2 (service 1) decreases. Since both services are same, except for the degreee of congestion, the two services compete with each other as if they were substitutes in this case.

Next, it is of interest to examine the signs of the cross derivatives of the user demand functions. The demand of user group 1 increases as either service 1's price or service 2's price decreases. The demand of user group 2 increases as service 1's price decreases, although user group 2 consumes only service 2. This is because, as a result of a decrease in service 1's price, some users who purchase service 2 switch to service 1, resulting in a decrease in service

2's congestion level. Hence user group 2 incurs a lower user cost than before, even with a fixed price for service 2. The effect of a price change of service 2 on the demand of user group 2 is indeterminate. A price increase of service 2 can lead to an increase in user group 2's demand under certain conditions. This is likely to occur when the demand elasticity of user group 1 and the congestion coefficient are high. The economic reasoning is as follows. As service 2's price increases, some of the users in user group 1 who consume service 2 turn to service 1. This results in a decrease in the congestion level of service 2. If a decrease in the congestion level of service 2 compensates for the price increase, user group 2 finds it has a lower user cost for servie 2 as a result of a price increase of service 2, thus increasing user group 2's demand for service 2.

Finally the effects of price changes on total market demand are examined. As service 1's price increases, total demand decreases. But an increase in service 2's price can result in an increase in total demand of $b_2\alpha(w^2-2w^1)-b_1w^1\alpha>0$. This property may lead to the case where a total surplus maximizer sets service 1's price below marginal cost. Therefore cross subsidization may exist under total surplus maximization.

Case 3: User group 1 consumes service 1, and user group 2 consumes service 2.

The demand conditions are

$$u^1=p_1+w^1\alpha x_1 \leqq p_2+w^1\alpha x_2 \qquad (31)$$

$$u^2=p_2+w^2\alpha x_2 \leqq p_1+w^2\alpha x_1 \qquad (32)$$

$$x_k=a_k-b_ku^k,\ k=1,\ 2 \qquad (33)$$

The market demand functions, x_1 and x_2, can be expressed in terms of prices of substituting u^1 and u^2 from Eqs. (31) and (32) into Eq. (33) and solving for x_1 and x_2, respectively:

$$x_1=q^1=(1+b_1w^1\alpha)^{-1}(a_1-b_1p_1) \qquad (34)$$

$$x^2=q^2=(1+b_2w^2\alpha)^{-1}(a_2-b\ p_2) \qquad (35)$$

Define R_3 ad the set of prices for which this case can occur., then we can find the set R_3 which satisfies the constraints (31), (32) and the nonnegativity of x_1 and x_2.

Case 4: User group 2 consumes both service 1 and service 2, and user group 1 consumes only service 1.

User group 2 incurs the same user cost for both services, and user group 1 incurs a least user cost for service 1. The corresponding demand equilibrium conditions are

$$u^1=p_1+w^1\alpha x_1 \leqq p_2+w^1\alpha x_2 \qquad (36)$$

$$u^2=p_2+w^2\alpha x_2=p_1+w^2\alpha x_1 \qquad (37)$$

$$x_1+x_2=\sum_{k=1}^{2}(a_k-b_ku^k) \qquad (38)$$

Substituting u^1 and u^2 from Eqs. (36) and (37) into Eq. (38) and using the relationship between x_1 and x_2 from Eq. (37), we obtain the market demand functions, x_1 and x_2, in items of prices, p_1 and p_2:

$$x_1 = (2 + b_1 w^1 \alpha + b_2 w^2 \alpha)^{-1} \left[(a_1 + a_2) - \frac{1 + b_1 w^2 \alpha + b_2 w^2 \alpha}{w^2 \alpha} p_1 + \frac{1}{w^2 \alpha} p_2 \right] \quad (39)$$

$$x_2 = (2 + b_1 w^1 \alpha + b_2 w^2 \alpha)^{-1} \times$$

$$\times \left[(a_1 + a_2) + \frac{1 + b_1 \alpha (w^1 - w^2)}{w^2 \alpha} p_1 - \frac{1 + b_1 w^1 \alpha + b_2 w^2 \alpha}{w^2 \alpha} p_2 \right] \quad (40)$$

Substituting x_1 and x_2 from Eqs. (39) and (40) into Eqs. (36) and (37) yields

$$u^1 = (2 + b_1 w^1 \alpha + b_2 w^2 \alpha)^{-1}$$

$$\times \left[(a_1 + a_2) w^1 \alpha + \left(2 + b_2 \alpha (w^2 - w^1) - \frac{w^1}{w^2}\right) p_1 + \left(\frac{w^1}{w^2}\right) p^2 \right] \quad (41)$$

$$u^2 = (2 + b_1 w^1 \alpha + b_2 w^2 \alpha)^{-1} \left[(a_1 + a_2) w^2 \alpha + (1 + b_1 \alpha (w^1 - w^2) p_1 + p_2 \right] \quad (42)$$

Substituting u^1 and u^2 from Eqs. (41) and (42) into Eq. (11) yields

$$q^1 = (2 + b_1 w^1 \alpha + b_2 w^2 \alpha)^{-1} \times \left[2 a_1 + a_1 b_2 w^2 \alpha - a_2 b_1 w^1 \alpha \right.$$

$$\left. - b_1 \left(b_2 \alpha (w^2 - w^1) + 2 - \frac{w^1}{w^2}\right) p_1 - b_1 \frac{w^1}{w^2} p^2 \right] \quad (43)$$

$$q^2 = (2 + b_1 w^1 \alpha + b_2 w^2 \alpha)^{-1}$$

$$\times \left[2 a_2 + a_2 b_1 w^1 \alpha - a_1 b_2 w^2 \alpha - b_2 \left[1 + b_1 \alpha (w^1 - w^2) \right] p_1 - b_2 p_2 \right] \quad (44)$$

The demand for service 1 by user group 2, q_1^2, can be determined by subtracting x_1 from q^1.

Finally, the total market demand x can be obtained by summing x_1 and x_2 or, equivalently q^1 and q^2:

$$x = (2 + b_1 w^1 \alpha + b_2 w^2 \alpha)^{-1} \left[2 (a_1 + a_2) + \frac{b_1 w^1 \alpha - 2 b_1 w^2 \alpha - b_2 w^2 \alpha}{w^2 \alpha} p_1 \right.$$

$$\left. - \frac{b_1 w^1 \alpha + b_2 w^2 \alpha}{w^2 \alpha} p_2 \right] \quad (45)$$

Define R_4 as the set of prices for which this case can occur, then we can find the set R_4 which satisfies constraint (36), and the nonnegativity of the user demands and market demands such as q^1, q^2, q_1^2, x_1 and x_2.

4. Properties of the Demands for Congested Services

A number of interesting observations can be made from examining the structure of market demand functions and user demand functions.

The effect of price changes on the market demand for each service is considered first (see Table 1). The market demand for each service decreases as its price increases. The market demand for the high-priced (less-congested) service may increase as a result of the decrease in the price of the more-congested service. In terms of cross-elasticity, this corresponds to the case where the services are complements. However, a more-congested service acts always as a substitute for a less-congested service; the demand for a more-congested service increases as the price of a less-congested service decreases.

The effect of price changes on the demand by each user group is considered next (see Table 2). The quantity demanded by users with a high time value may increase as the price of the more-congested service increase. This is because as the price of the more-congested service increases, the demand by users with a low value of the decreases. The decrease of the demand by these users reduces the equilibrium level of congestion. The user cost incurred by users with a high value of time may therefore decrease, and consequently their demand can increase in spite of the increase in the price of the more-congested service.

In summary, the market demand function for each service has been derived as a function of prices, user time values and other user demand characteristics. Since the market demand for each service is a result of user self-selection process, for any given set of services, prices set by providers, user time values, and user demands, the corresponding demand faced by each provider can be uniquely obtained.

Table 1. The Effects of Price Changes on Market Demands

	x_1		x_2		$x(=x_1+x_2)$	
	$\partial x_1/\partial p_1$	$\partial x_1/\partial p_2$	$\partial x_2/\partial p_1$	$\partial x_2/\partial p_2$	$\partial x/\partial p_1$	$\partial x/\partial p_2$
Case 1	−	0	0	0	−	0
Case 2	−	+	+	−	−	±
Case 3	−	0	0	−	−	−
Case 4	−	+	±	−	−	−

Table 2. The Effects of Price Changes on User Demands

	q^1		q^2	
	$\partial q^1/\partial p_1$	$\partial q^1/\partial p_2$	$\partial q^2/\partial p_1$	$\partial q^2/\partial p_2$
Case 1	−	0	−	0
Case 2	−	−	−	±
Case 3	−	0	0	−
Case 4	−	−	±	−

5. An Example Comprising Uniform vs. Nonuniform Pricing Policies

In this example, a provider of service, such as a university, is providing a service, such as timeshared computing service, initially to a user population at zero price. It is assumed that the user population is divided into two user groups (possibly users with an immediate deadline and users with a more remote deadline for completing their work), each of which has different time values. In this case, let $w^1=1$ and $w^2=2$. The user demand and congestion functions are assumed to be linear as follows:

$$q^1 = 10 - u^1$$
$$q^2 = 20 - u^2$$
$$\tau_1(x) = \alpha_1 x_1$$

where α_1 denotes the congestion coefficient of an existing facility, and is assumed to be 1. The cost per unit time of providing these services is assumed to depend on the value of the congestion coefficient of the system. This cost is assumed to be $(10/\alpha_1)$. Since the services are provided at zero price, the university incurs a net cost of 10 in providing the services in this case. Given the user demands of the two user groups and the existing system, the equilibrium congestion level and user demands are (see case 1 above)

$$\tau_1(x_1) = 7.5$$
$$q^1 = 2.5 \qquad u^1 = 7.5$$
$$q^2 = 5 \qquad u^2 = 15$$

The university feels that the level of congestion is too high and decides to increase the budget from 10 to 15. With the increased budget, two alternatives are considered: the existing zero price policy and a nonzero price policy. Under the zero price policy, a new facility with a congestion coefficient 2 will be made available, in addition to the existing facility, and the net cost involved in providing the services will be 15. Under this zero price policy, the equilibrium congestion level for each facility and user demands are (see the case 4 above)

$$x_1 = 6.68$$
$$x_2 = 3.33$$
$$\tau_1(x_1) = x_1 = 6.67$$
$$\tau_2(x_2) = 2x_2 = 6.67$$
$$q^1 = 3.33 \qquad u^1 = 6.67$$
$$q^2 = 6.67 \qquad u^2 = 13.33$$

where x_1 denotes the quantity of services provided from the existing facility, x_2 denotes the quantity of services provided from the new facility, τ_1 denotes the equilibrium congestion level for the existing facility, and τ_2 denotes the equilibrium congestion level for the new system. The congestion coefficient for each service is different. However, the equilibrium congestion level (unit cost) for each service is the same. The user cost has been reduced for both user groups and the quantity of service provided to each group has been increased, as a result of the increased budget allocation.

We now consider a nonzero price policy. Under this policy the university will charge for services from the new facility and use the revenues collected from the new facility to cover some of the cost involved in providing the services from the new facility. The university decides to set the congestion coefficient for the new facility and the price for services from the new facility to the values that maximize the user surplus subject to the budget allocated:

$$\max_{p_2, \alpha_2} \int_{u^1}^{\infty} q^1(v)dv + \int_{u^2}^{\infty} q^2(v)dv$$

subject to

$$p_2 x_2 = (10/\alpha_2) - 5$$
$$u^1 = p_1 + w^1 \alpha_1 x_1$$
$$u^2 = p_2 + w^2 \alpha_2 x_2$$

where x_1 is the demand for services from the existing facility, x_2 is the demand for services from the new facility, and p_1, α_1, w^1, and w^2 are set to 0,1,1, and 2 respectively.

Solving this optimization problem for p_2 and α_2, we obtain the congestion coefficient for the new facility, the price for services from the new facility, equilibrium congestion level for each facility, user demand of each user group:

$$p_2 = 4.15$$
$$\alpha_2 = 0.19$$
$$\tau_1(x_1) = 5$$
$$\tau_2(x_2) = 2.18$$
$$q^1 = 5 \qquad u^1 = 5$$
$$q^2 = 11.47 \qquad u^2 = 8.53$$

Again, both user groups have reduced user costs and increased consumption levels in comparison with previous cases. But the budget allocation is still the same as in the preceding case.

The cost involved in providing services for the new facility will be $52.63 = (10/\alpha_2)$, which is the sum of the revenues collected from the facility, 47.63, and the increased budget, 5. The use of the revenues collected from the high time value users has permitted the university to acquire a facility with a substantially higher capacity than was possible before. The comparisons between the two alternatives are summarized in Table 3. From Table 3, both user groups are better off under the nonzero price policy. This example shows that total welfare is increased simply by charging for services from the new facility and allowing the users to self-select which facility to use. Total welfare is defined here as the difference between' user surplus and the university's net cost.

Since the university incurs the same net cost under the two new system alternatives, the comparison of these two alternatives is appropriate. The users with low time values still obtain service at zero price, and their time costs are reduced, because the nonzero price facility draws off high time value users. The users with high time values must pay a nonzero price, but their total user costs are reduced.

In this case, the higher priced product attracts the user group with the higher value of time, and a form of discriminatory pricing is made possible in which users self-select themselves. Benefits to both the user group with the low user time value and the user group with the high user time value are greater for positive pricing. It appears that there may be opportunities for increasing both user benefits and provider profits in congested systems serving user groups with different time values, simply by offering the same service through multiple sources at different prices.

Table 3 The Comparison of a Zero Price Policy and an Nonzoro Price Policy

	Existing System	New System	
	Zero Price Policy	Zero Price Policy	Nonzero Price Policy
p_1	0	0	0
p_2	–	0	4.15
α_1	1	1	1
α_2	–	2	0.19
x_1	7.5	6.67	5
x_2	–	3.33	11.47
$\tau_1(x_1)$	7.5	6.67	5
$\tau_2(x_2)$	–	6.67	2.18
u^1	7.5	6.67	5
u^2	15	13.33	8.53
University Net Cost	10	15	15
Welfare	21.25	40.58	141.98

6. Conclusions

In a congested market where users differ in their values of time and have linear demand curves, a method for aggregating user demand has been developed and used to obtain market demands for services from different sources as a function of prices of services set by providers. It can be shown that, for any set of prices, user time values, and user demand characteristics such as demand elasticity, the resulting market demand function for each service is a continuous function of prices and is piecewise differentiable over the whole price space. For any set of services the method developed here can be used to predict the equilibrium levels of congestion, the amount of service consumed by each user group, the amount of each service provided to each user group, and so on.

An examination of the cross elasticity of market demand revealed the special features of a market with congestion, which are not observed in a market in which product qualities are set solely by producers. The market demand for a high-priced (less congested) service may increase (decrease) as the price of a low-price (more congested) service decreases (increases), when the demands by users with a low value of time are more elastic or the difference between the two user time values is large. In this case a high-priced service is seen by suppliers as a complementary service of a low-priced service. However, the market demand for a low-priced service always increase (decreases) as the price of a high-priced service increases (decreases): a low-priced service is always seen as a substitute service of a high-priced service. The policy implication of this property is of significance; a profit-maximizing monopolist may engage in a cross subsidy between the two services by setting the price of a highly-congested service below the marginal cost.

Another important feature of such a market is that the increase in the price of a less-congested service may result in the increase of the total demand for the service. For example, consider the case in which government provides pure public services subject to congestion. Simply by charging a price for a less-congested service, the revenues provided by such charges enable the government to partly avoid the conflict between efficiency and benefit finance which it confronts with respect to the congested services.

Finally, comparisons between uniform pricing and nonuniform pricing policies are made. A specific numerical example is given to show that a nonuniform pricing policy may be Pareto superior to a uniform pricing policy. This suggests that a nonuniform pricing scheme for a congested service may improve the utilization of the congested system.

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