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A Study on Estimation of Parameters in Bivariate Exponential Distribution

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ABSTRACT

Estimation for the parameters of a bivariate exponential (BVE) model of Marshall and Olkin (1967) is investigated for the cases of complete sampling and time-truncated parallel sampling. Maximum likelihood estimators, method of moment estimators and Bayes estimators for the parameters of a BVE model are obtained and compared with each other. A Monte Cario simulation study for a moderate sized samples indicates that the Bayes estimators of parameters perform better than their maximum likelihood and method of moment estimators.

1. Introduction

In many reliability analyses, the estimation of the reliability for multi-component system and associated life testing problem has investigated by Rutemiller (1966), Bhattacharya (1967), Sinha and Guttman (1976), Zack (1977), Basu and El Mawaziny (1978), Chao and Hwang (1983) and so on, under the assumption of stochastic independence among the component of the system. But in many practical situations although the assumption that the components of system have underlying exponential distributions may be reasonable, the assumption of stochastic independence among the components may not. Hence it is more realistic to assume some form of dependence among the components of the system. This dependence among the components arise from common environmental stresses and shocks, from component depending on common sources of power, and so on.

Several bivariate models based on exponential distributions have been derived. The bivariate exponential model of Freund (1961) is based on the joint system reliability of two components which

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initially are independently on test with exponential distributions with parameters α and β respectively. Failure of one components reduces the additional mean life of the remaining component by increasing either α to α' or β to β'. This model does not have exponential marginals, but has the loss of memory property. Moreover, Freund's model does not allow for the simultaneous failures. The bivariate exponential (BVE) model of Marshall and Olkin (1967) is based a fatal shock model characterization, which is suggestive of potential applications. This model is applicable as a failure model for such systems when there exists positive probability of simultaneous failure of exponential components. Marshall and Olkin's model is not absolutely continuous, but have exponential marginals and the loss of memory property. Downton (1970) proposed a bivariate exponential model to describe the relationship between the lifetimes of two components which are subject to random shock and the number of shock needed to lead a failure follows a bivariate geometric distribution. This model has exponential marginals, but does not have the loss of memory property. Hawkes (1971) proposed a more general successive damage model than Downton's model. The distributions of both Downton and Hawkes are absolutely continuous. Block and Basu (1974) proposed an absolutely continuous bivariate exponential (ACBVE) model that turned out to be the absolutely continuous part of the Marshall and Olkin's BVE model, as well as a variant of the Freund's model. To obtain an ACBVE model with the loss of memory property they did not assume exponential marginals.

Among several bivariate exponential models, since the BVE model of Marshall and Olkin (1967) have exponential marginals and the loss of memory property, this model can play a central role in the reliability of the system with dependent components. Moreover, this model is appropriate for the case of simultaneous failures.

In this paper we consider the estimation for the parameters of BVE model of Marshall and Olkin (1967). In section 2, we introduce some properties of BVE model and obtain the M.L.Es, M.M.Es and Bayes estimators of parameters of BVE model in the case of complete sampling. In section 3, we obtain the M.L.Es and Bayes estimators of parameters of BVE model in the case of time-truncated parallel sampling. In section 4, we investigate the relative performance of M.L.Es, M.M.Es and Bayes estimators of parameters for a moderate sized samples through Monte Carlo simulation in both sampling cases.

Finally, in section 5, conclusion and some suggestions for further research are given.

2. Estimation for the Parameters of BVE Model in the Case of Complete Sampling

2.1 Derivation of Model and its Properties

Marshall and Olkin (1967) proposed a BVE model for a system where the lifetimes of components may be dependent due to shocks affecting two components simultaneously.

Let T_1 and T_2 be the lifetimes of components 1 and 2, respectively, and $\{Z_i(t),\ t \geqq 0\}$, $i=0, 1, 2$, be three independent poisson processes with corresponding failure rates

$$\lambda_i,\ i=0, 1, 2,\quad \lambda \in \Lambda,$$

where

$$\underline{\lambda}=(\lambda_1,\ \lambda_2,\ \lambda_0),$$

and

$$\Lambda=\{\underline{\lambda} | 0 \leqq \lambda_i < \infty,\ \ i=0,\ 1,\ 2;\ \lambda_j+\lambda_0>0,\ j=1,\ 2\}$$

A shock in the $Z_i(t)$ is selectively fatal to component i, $i=1,2$, while a shock in the $Z_0(t)$ is simultaneously fatal to both components. Let U_i represent the times to the first shocks in the $Z_i(t), i=0,1,2,$ processes, respectively. Then the joint reliability of (T_1, T_2), $T_i=\min(u_0, u_i)$, $i=1,2$, is as follows:

(2.1) $$\bar{F}(t_1, t_2)=Pr(T_1>t_1, T_2>t_2)$$

$$=\exp[-\lambda_1 t_1-\lambda_2 t_2-\lambda_0 \max(t_1, t_2)].$$

when $\underline{T}=(T_1, T_2)$ has the three parameter BVE distribution of (2.1), we denote $\underline{T}\sim \mathrm{BVE}(\lambda_1, \lambda_2, \lambda_0)$.

Let E_2 denote two dimensional Euclidean space and

$E_2^+=\{\underline{t}\in E_2: t_i>0, i=1, 2\}$. For $\underline{t}\in E_2^+$, define

(2.2) $$t_{(1)}(\underline{t})\equiv t_{(1)}=\min(t_1, t_2),$$

$$t_{(2)}(\underline{t})\equiv t_{(2)}=\max(t_1, t_2),$$

$$V_i(\underline{t})\equiv V_i=\begin{cases}1 & \text{if } t_i<t_{(2)}\\ 0 & \text{elsewhere,}\end{cases}$$

and

$$y(\underline{t})\equiv y=t_{(2)}-t_{(1)}.$$

Then, as noted by Proschan and Sullo (1974), the following facts are obtained:

(2.3)
(1) T_i is exponential with parameter $\gamma_i=\lambda_i+\lambda_0$, $i=1, 2$,

(2) $T_{(1)}$ is exponential with parameter $\lambda=\lambda_1+\lambda_2+\lambda_0$,

(3) $T_{(1)}$ is independent of $\underline{V}=(V_1, V_2)$, and Y,

(4) $(V_1, V_2, 1-V_1-V_2)$ is trinomial $\left(1; \frac{\lambda_1}{\lambda}, \frac{\lambda_2}{\lambda}, \frac{\lambda_0}{\lambda}\right)$,

(5) Given that $V_1=1(V_2=1)$, Y is exponential with parameter $\gamma_2(\gamma_1)$.

Let complete sampling for the BVE model be available. Then the sample consists of observations on random variable $(T_{(1)}, \underline{v}, y)$. Hence the probability density function ($p.d.f.$) of $(T_{(1)}, \underline{v}, y)$ is derived as follows:

(2.4) $$g(t_{(1)}, \underline{v}, y)=g_1(t_{(1)}, \underline{v})\cdot g_2(y|v),$$

where

$$g_1(t_{(1)}, \underline{v})=\lambda_1^{v_1}\lambda_2^{v_2}\lambda_0^{1-v_1-v_2}\exp[-\lambda t_{(1)}],$$

and

(2.5) $$g_2(y|v)=\{\gamma_2\exp[-\gamma_2 y]\}^{v_1}\{\gamma_1\exp[-\gamma_1 y]\}^{v_2}\{I(y=0)\}^{1-v_1-v_2}$$

Let (T_{1j}, T_{2j}), $j=1, \ldots, n$, be a random sample from BVE $(\lambda_1, \lambda_2, \lambda_0)$. Then the likelihood function of the complete sampling case is given by

$$(2.6) \qquad L(\underline{\lambda}) = \prod_{j=1}^{n} g(t_{(1)j}, \underline{v}_j, y_j)$$

$$= (\lambda_1 \gamma_2)^{n_1} (\gamma_1 \lambda_2)^{n_2} \lambda_0^{n_0} \exp[-\lambda \sum t_{(1)j} - \gamma_2 \sum v_{1j} y_j - \gamma_1 \sum v_{2j} y_j]$$

$$\text{for} \quad (\lambda_1, \lambda_2, \lambda_0) \in \Lambda,$$

where

$$n_1 = \sum v_{1j}, \; n_2 = \sum v_{2j} \text{ and } n_0 = \sum (1 - v_{1j} - v_{2j}) = n - n_1 - n_2$$

2.2 The M.L.Es for Parameters of Complete Sampling Case

Equating the first partial derivatives of log $L(\underline{\lambda})$ to be zero, we obtain the likelihood equations:

$$(2.7) \qquad \frac{n_1}{\lambda_1} + \frac{n_2}{\gamma_1} = \sum t_{(1)j} + \sum v_{2j} y_j,$$

$$\frac{n_1}{\gamma_2} + \frac{n_2}{\lambda_2} = \sum t_{(1)j} + \sum v_{1j} y_j,$$

$$\frac{n_2}{\gamma_1} + \frac{n_1}{\gamma_2} + \frac{n_0}{\lambda_0} = \sum t_{(1)j} + \sum v_{1j} y_j + \sum v_{2j} y_j$$

Then the likelihood equations in (2.7) are equivalent to those obtained by Bemin, Bain and Higgins (1972) and Bhattacharya and Johnson (1971). In a various cases, the M.L.Es of $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_0)$ are give by:

(2.8) (1) For $n_0, n_1, n_2 > 0$, the M.L.E of $\underline{\lambda}$ is unique and is the unique root of (2.7).

(2) For $n_0 = 0$, $n_1 > 0$, $n_2 > 0$, the M.L.E. of $\underline{\lambda}$ is unique and is given by:

$$\hat{\lambda}_1 = \frac{n}{\sum t_{1j}}, \quad \lambda_2 = \frac{n}{\sum t_{2j}} \text{ and } \lambda_0 = 0$$

(3) For $n_0 = 0$ and either $n_1 = 0$ or $n_2 = 0$, the M.L.E. of λ is exists but is not unique.

(4) For $n_0 > 0$ and one or both of n_1 and n_2 are 0, the M.L.E. of $\underline{\lambda}$ does not exist within Λ.

In the BVE $(\lambda_1, \lambda_1, \lambda_0)$ with identical marginals, the M.L.Es of λ_1 and λ_0 are easily obtained as follows:

(2.9) (1) For $0 < n_0 < n$, the M.L.E. of (λ_1, λ_0) is uniquely given by

$$\lambda_0=\frac{1}{2S_1S_2}\ \{[n^2(S_2-S_1)^2+4n_0(2n-n_0)S_1S_2]^{\frac{1}{2}}-n(S_2-S_1)\},$$

$$\lambda_1=(n-n_0)\hat{\lambda}_0[n_0+\hat{\lambda}_0S_1]^{-1},$$

where

$$S_1=\sum t_{(1)j} \text{ and } S_2=\sum t_{(2)j}$$

(2) For $n_0=0$, the M.L.E. of (λ_1, λ_0) is uniquely given by

$$\hat{\lambda}_0=0 \text{ and } \hat{\lambda}_1=\frac{2n}{S_1+S_2}$$

(3) For $n_0=n$, the M.L.Es of (λ_1, λ_0) does not exist within the parameter space Λ .

2.3 The M.M.Es for Parameters of Complete Sampling Case

From (2.3) and $\rho=P[T_1=T_2]=\frac{\lambda_0}{\lambda}$, Bemis, Bain and Higgins (1972) obtained the M.M.E. for $(\lambda_1, \lambda_2, \lambda_0)$ as follows:

(2.10) $$\hat{\hat{\lambda}}_1=[\frac{n}{\sum t_{1j}}-\frac{n_0}{\sum t_{2j}}]/(1+\frac{n_0}{n}),$$

$$\hat{\hat{\lambda}}_2=[\frac{n}{\sum t_{2j}}-\frac{n_0}{\sum t_{1j}}]/(1+\frac{n_0}{n}),$$

and

$$\hat{\hat{\lambda}}_0=n_0[\frac{1}{\sum t_{1j}}+\frac{1}{\sum t_{2j}}]/(1+\frac{n_0}{n}).$$

In the BVE $(\lambda_1, \lambda_1, \lambda_0)$ with identical marginals, we know that $T_{(1)}$ is exponential with parameter $\lambda=2\lambda_1+\lambda_0$ and $\rho=\frac{\lambda_0}{2\lambda_1+\lambda_0}$. Hence we obtain the M.M.Es of λ_1 and λ_0 as follows:

(2.11) $$\hat{\hat{\lambda}}_1=\frac{n-n_0}{2\sum t_{(1)j}} \text{ and } \hat{\hat{\lambda}}_0=\frac{n_0}{\sum t_{(1)j}}$$

2.4 The Bayes Estimators for Parameters of Complete Sampling Case

Now, we consider the Bayes estimators of $\underset{\sim}{\lambda}$ for BVE $(\lambda_1, \lambda_1, \lambda_0)$ with identical marginals. To simplify notation, let

$$n_3=n_1+n_2$$

$$\tau_1=2\sum t_{(1)j}+\sum(V_{1j}+V_{2j})y_j,$$

and

$$\tau_0 = \sum t_{(1)j} + \sum (v_{1j} + v_{2j}) y_j = \tau_1 - \sum t_{(1)j}$$

Then in the BVE $(\lambda_1, \lambda_1, \lambda_0)$ model the likelihood function is given by

$$L(\lambda_1, \lambda_0) = \sum_{l=0}^{n_3} \binom{n_3}{l} \lambda_1^{2n_3 - l} \lambda_0^{n_0 + l} \exp[-\lambda_1 \tau_1 - \lambda_0 \tau_0]. \tag{2.12}$$

We assume a quadratic loss function defined by

$$L(\underline{\lambda}, \underline{\lambda}^*) = (\underline{\lambda} - \underline{\lambda}^*)'(\underline{\lambda} - \underline{\lambda}^*),$$

and the prior distribution for $\underline{\lambda} = (\lambda_1, \lambda_1, \lambda_0)$ is vague prior as follows:

$$g(\lambda_1, \lambda_0) \propto \frac{1}{\lambda_1^{c_1} \lambda_0^{c_0}}, \quad \lambda_1, \lambda_0 > 0, \ c_1 > 0, \ c_0 > 0. \tag{2.13}$$

Then the joint posterior distribution for λ_1 and λ_0 is given by

$$\begin{aligned} \Pi(\lambda_1, \lambda_0 | d) &= k \, g(\lambda_1, \lambda_0) \cdot L(\lambda_1, \lambda_0 | d) \\ &= k \sum_{l=0}^{n_3} \binom{n_3}{l} \lambda_1^{2n_3 - c_1 - l} \lambda_0^{n_0 + l - c_0} \\ &\quad \cdot \exp[-\lambda_1 \tau_1 - \lambda_0 \tau_0], \quad 0 < \lambda_i < \infty, \ i = 0, 1, \end{aligned} \tag{2.14}$$

where

$$K^{-1} = \sum_{l=0}^{n_3} \binom{n_3}{l} \frac{\Gamma(2n_3 - c_1 - l + 1) \cdot \Gamma(n_0 + l - c_0 + 1)}{\tau_1^{2n_3 - c_1 - l + 1} \cdot \tau_0^{n_0 + l - c_0 + 1}}$$

and

$\underline{d} = (\tau_1, \tau_0, n_3)$ is a set of minimal sufficient statistics for BVE $(\lambda_1, \lambda_1, \lambda_0)$.

Let $\pi_1(\lambda_1)$ and $\pi_0(\lambda_0)$ be the marginal posterior p.d.f. of λ_1 and λ_0, respectively. Integrating out λ_0 in (2.14), the marginal posterior p.d.f. of λ_1 is given by

$$\pi_1(\lambda_1) = K \sum_{l=0}^{n_3} \binom{n_3}{l} \lambda_1^{2n_3 - c_1 - l} \exp[-\lambda_1 \tau_1] \frac{\Gamma(n_0 + l - c_0 + 1)}{\tau_0^{n_0 + l - c_0 + 1}}, \quad 0 < \lambda_1 < \infty \tag{2.15}$$

The Bayes estimator of λ_1 is the posterior mean of λ_1 under the quadratic loss function, therefore, we can easily obtain the Bayes estimator of λ_1 as follows:

$$\lambda_1^* = E_{\lambda 1}(\lambda_1) = K \sum_{l=0}^{n_3} \binom{n_3}{l} \frac{\Xi(2n_3 - l - c_1 + 2) \cdot \Gamma(n_0 + l - c_0 + 1)}{\tau_1^{2n_3 - l - c_1 + 2} \cdot \tau_0^{n_0 + l - c_0 + 1}} \tag{2.16}$$

where K is given in (2.14).
Similarly, the Bayes estimator of λ_0 under the quadratic loss function is given by

(2.17) $$\lambda_0^* = K \sum_{l=0}^{n_3} \binom{n_3}{l} \frac{\Gamma^{(2n_3-l-c_1+1)} \cdot \Gamma^{(n_0+l-c_0+2)}}{\tau_1^{2n_3-l-c_1+1} \cdot \tau_0^{n_0+l-c_0+2}}$$

Remark Since the induced family of distributions of the minimal sufficient statistics for BVE $(\lambda_1, \lambda_2, \lambda_0)$ is not complete, some difficulties are presented in establishing whether or not a minimum variance unbiased estimator for $\underline{\lambda}$ exists, as noted by Bhattacharya and Johnson (1971).

3. Estimation for the Parameters of BVE Model in the Case of Time-Truncated Sampling

It is desirable to terminate tests at some preassigned time. In this section we assume that testing will be terminated at time

(3.1) $$X_T = T_{(1)(n)} + x_0$$

where

x_0 is preassigned,

and

$$T_{(1)(n)} = \max\{T_{(1)j},\ j = 1, \ldots, n\}.$$

Definition The time-truncated parallel sampling for BVE model is the procedure which is to wait until at least one component in each parallel system on test has failed and then to continue testing for an additional predetermined time x_0. Under the time-truncated parallel sampling the identity of the component which fails first and the times of both failures are observable.

Let

$$A = \{j: T_{(2)j} \leqq X_T\} = \{j: Y_j \leqq X_T - T_{(1)j}\},$$

and

$$n_i(A) = \sum_A v_{ij},\ i = 1, 2, \quad n_0(A) = n_0$$

Define

(3.2) $$Y_j^* = \begin{cases} Y_j, & if \quad Y_j \in A, \\ X_T - T_{(1)j}, & if \quad Y_j \in A^c, \quad j = l, \ldots,, n. \end{cases}$$

For each Y_j^* depends on the $T_{(1)r}$, $r = 1, \ldots, n$, only through $T_{(1)(n)} - T_{(1)j}$. Hence the conditional p.d.f. of Y_j^* given $\{T_{(1)j}, v_j, \quad j = \ \ldots, n\}$ e ;

(3.3) $$g_2^*(y_j | t_{(1)j}, \underline{v}_j) = \begin{cases} g_2(y_j | t_{(1)j}, \underline{v}_j) & \text{if } y_j \in A, \\ Pr(Y_j > X_T - t_{(1)j} | v_j) & \text{if } y_j \in A^c, \end{cases}$$

where

$$g_1(y_j | t_{(1)j}, \underline{v}_j) = g_2(y_j | v_j) \quad \text{in } (2.5),$$

and

$$Pr(Y_j > X_T - t_{(1)j} | \underline{v}_j) = \exp[-(\gamma_2 v_{1j} + \gamma_1 v_{2j})(X_T - t_{(1)j})].$$

Hence, the p.d.f. of $(T_{(1)}, \underline{v}, y^*)$ is obtained as follows:

$$(3.4) \qquad f(t_{(1)}, \underline{v}, y^*) = g_1(t_{(1)}, \underline{v}) \cdot g_2^*(y | t_{(1)}, \underline{v}),$$

where

$$g_1(t_{(1)}, v) = \lambda_1^{v_1} \lambda_2^{v_2} \lambda_0^{1-v_1-v_2} \exp[-\lambda t_{(1)}] \quad \text{in } (2.4),$$

and

$g_2^*(y | t_{(1)}, \underline{v})$ is given by (3.3).

Let (T_{1j}, T_{2j}), $j = 1, \dots, n$, be a random sample from BVE $(\lambda_1, \lambda_2, \lambda_0)$ model. Then the likelihood function of the time-truncated parallel samples is given by

$$(3.5) \qquad L_T(\lambda) = \prod_{j=1}^{n} f(t_{(1)}, \underline{v}, y^*)$$

$$= \lambda_1^{n_1} \lambda_2^{n_2} \lambda_0^{n_0} \exp[-\lambda \Sigma t_{(1)j}] \cdot \gamma_2^{n_1(A)} \gamma_1^{n_2(A)}$$

$$\cdot \exp[-\gamma_2 \sum_A v_{1j} y_j - \gamma_1 \sum_A v_{2j} y_j]$$

$$\cdot \exp[-\sum_{A^c} (\gamma_2 v_{1j} + \gamma_1 v_{2j})(X_T - t_{(1)j})].$$

3.1 The M.L.Es for Parameters of Time-Truncated Parallel Sampling Case

Equating the first partial derivatives of log $L_T(\underline{\lambda})$ to be zero, we obtain the likelihood equations as follows:

$$(3.6) \qquad \frac{n_1}{\lambda_1} + \frac{n_2(A)}{\gamma_1} = \sum_{j=1}^{n-n_2(A^c)} t_{1(j)} + n_2(A^c) \cdot X_T,$$

$$\frac{n_2}{\lambda_2} + \frac{n_1(A)}{\gamma_2} = \sum_{j=1}^{n-n_1(A^c)} t_{2(j)} + n_1(A^c) \cdot X_T,$$

and

$$\frac{n_2(A)}{\gamma_1} + \frac{n_1(A)}{\gamma_2} + \frac{n_0}{\lambda_0} = \sum_{j=1}^{n-n_1(A^c)-n_2(A^c)} t_{(2)(j)} + [n_1(A^c) + n_2(A^c)] \cdot X_T$$

which are not explicitly solvable.

In the BVE $(\lambda_1, \lambda_1, \lambda_0)$ with identical marginals, the M.L.Es of λ_1 and λ_0 are obtained as follows:

(3.7) (1) For $0 < n_0 < n$, the M.L.Es of λ_1 and λ_0 are uniquely given by

$$\hat{\lambda}_0 = (2a)^{-1}[(b^2+4ac)^{\frac{1}{2}} - b],$$

and

$$\hat{\lambda}_1 = \hat{\lambda}_0 \cdot (n_1 + n_2)/[n_0 + \hat{\lambda}_0 \Sigma t_{(1)j}],$$

where

$$a = \Sigma t_{(1)j}\left[\sum_{j=1}^{n-n_1(A^c)-n_2(A^c)} t_{(2)j} + (n_1(A) + n_2(A)).\ X_T\right],$$

$$b = [-n_0 + n_1(A) + n_2(A)]\sum t_{(1)j}$$

$$+\, n\left[\sum_{j=1}^{n-n_1(A)-n_2(A)} t_{(2)j} + (n_1(A) + n_2(A)).\ X_T\right],$$

and

$$c = n_0(n + n_1(A) + n_2(A)).$$

(2) For $n_0 = 0$, the M.L.Es of λ_1 and λ_0 are uniquely given by $\hat{\lambda}_0 = 0$,

and

$$\hat{\lambda}_1 = [n_1 + n_2 + n_1(A) + n_2(A)]/[\sum t_{(1)j} + \sum_{j=1}^{n-n_1(A)-n_2(A)} t_{(2)j}$$

$$+ (n_1(A^c) + n_2(A^c)).\ X_T].$$

(3) For $n_0 = n$, the M.L.Es of λ_1 and λ_0 do not exist within the parameter space Λ .

Remark In the BVE $(\lambda_1, \lambda_1, \lambda_0)$ with identical marginals and the case time-truncated parallel sampling, we also have that $T_{(1)}$ is exponential with parameter $\lambda = 2\lambda_1 + \lambda_0$ and $\rho = \dfrac{\lambda_0}{2\lambda_1 + \lambda_0}$ Hence the M.M.Es of λ_1 and λ_0 are equivalent to $\hat{\hat{\lambda}}_1$ and $\hat{\hat{\lambda}}_0$ which are coincided with the complete sampling case.

3.2 The Bayes Estimators for Parameters Under the Time-Truncated Parallel Sampling

Now, we consider the Bayes estimator of $\underset{\sim}{\lambda}$ for BVE $(\lambda_1, \lambda_1, \lambda_0)$ with identical marginals. To simplify notation, let

(3.8) $n_3 = n_1 + n_2,$

$$n_3(A) = n_1(A) + n_2(A),$$

$$\tau_1^* = 2\Sigma t_{(1)j} + \sum_A (v_{1j} + v_{2j}) + \sum_{A^c} (v_{1j} + v_{2j})(X_T - t_{(1)j}),$$

and

$$\tau_0^* = \tau_1^* - \Sigma t_{(1)j} \cdot$$

Then in the BVE $(\lambda_1, \lambda_1, \lambda_0)$ model the likelihood function is given by

$$L_T(\lambda_1, \lambda_0) = \sum_{l=0}^{n_3(A)} \binom{n_3(A)}{l} \lambda_1^{n_3 + n_3(A) - l} \lambda_0^{n_0 + l} \exp[-\lambda_1 \tau_1^* - \lambda_0 \tau_0^*]. \tag{3.9}$$

We assume a quadratic loss function defined by

$$L(\underline{\lambda}, \underline{\lambda}^*) = (\underline{\lambda} - \underline{\lambda}^*)'(\underline{\lambda} - \underline{\lambda}^*),$$

and the prior distribution for (λ_1, λ_0) is vague prior given by

$$g(\lambda_1, \lambda_0) \propto \frac{1}{\lambda_1^{c_1} \lambda_0^{c_0}}, \quad \lambda_1, \lambda_0 > 0$$

where c_1 and c_0 are positive constants.

Let $\pi_T(\lambda_0, \lambda_1)$ be the joint posterior p.d.f. of λ_1 and λ_0, given by

$$\Pi_T(\lambda_0, \lambda_1) = K_T \sum_{l=0}^{n_3(A)} \binom{n_3(A)}{l} \lambda_1^{n_3 + n_3(A) - l - c_1} \lambda_0^{n_0 + l - c_0} \tag{3.10}$$

$$\cdot \exp[-\lambda_1 \tau_1^* - \lambda_0 \tau_0^*], \quad 0 < \lambda_1 < \infty, \quad 1 = 1, 2,$$

where

$$K_T^{-1} = \sum_{l=0}^{n_3(A)} \binom{n_3(A)}{l} \frac{\Gamma(n_3 + n_3(A) - l - c_1 + 1) \cdot \Gamma(n_0 + l - c_0 + 1)}{(\tau_1^*)^{n_3 + n_3(A) - l + c_1 + 1} (\tau_0^*)^{n_0 + l - c_0 + 1}}$$

Hence, under the quadratic loss function and vague prior distribution, we can easily obtain the Bayes estimators of λ_1 and λ_0 as follows:

$$\lambda_1^{**} = K_T \sum_{l=0}^{n_3(A)} \binom{n_3(A)}{l} \frac{\Gamma(n_3 + n_3(A) - l - c_1 + 2) \cdot \Gamma(n_0 + l - c_0 + 1)}{(\tau_1^*)^{n_3 + n_3(A) - l - c_1 + 2} (\tau_0^*)^{n_0 + l - c_0 + 1}} \tag{3.11}$$

and

$$\lambda_0^{**} = K_T \sum_{l=0}^{n_3(A)} \binom{n_3(A)}{l} \frac{\Gamma(n_0 + l - c_0 + 2) \cdot \Gamma(n_3 + n_3(A) - l - c_1 + 1)}{(\tau_0^*)^{n_0 + l - c_0 + 2} (\tau_1^*)^{n_3 + n_3(A) - l - c_1 + 1}}, \tag{3.12}$$

K_l is given by (3.10).

Remark. As $x_0 \to \infty$, (3.11) and (3.12) is equivalent to (2.15) and (2.16).

4. Empirical Comparison for a Moderate Sized Samples

In this section we investigate the relative performance a moderate sized samples through Monte Carlo study. The efficiency of the estimators is measured in terms of the ratio of the trace of the inverse of information matrix to the sum of the M.S.Es of the individual estimators. It is given by

$$\text{Eff}\cdot = tr(In^{-1})/\Sigma\text{MSE}(\hat{\lambda}_1).$$

All computation in this section are carried out on Cyber-170/815. For fixed n = 7, 10 and 20, estimates of the mean squared error are obtained from 4,000 simulated samples with $\lambda_1 = \lambda_2 = 1$, $\lambda_0 = 0.1$, 0.5, and 1.5. In each trial, two dependent exponential random variables of Marshall and Olkin's model are generated as follows:

Let three random variables u_0, u_1 and u_2 be independent exponential distribution with parameters λ_0, λ_1 and λ_2, respectively. Define T_1 and T_2 as follows:

$$T_1 = \min(u_0, u_1),$$

and

$$T_2 = \min(u_0, u_2).$$

Then (T_1, T_2) have the BVE $(\lambda_1, \lambda_2, \lambda_0)$ distribution given by (2.1).

The efficiencies for the estimators under the complete and time-truncated parallel sampling appear in Table 1 and 2, respectively.

From Table 1 and 2, we know the following facts:

(1) The Bayes estimators of parameters performs better than their M.L.Es and M.M.Es under both sampling, except for the case of n = 10, and 20, $\rho = 0.429$.

(2) In table 1, the efficiencies of M.L.Es and Bayes estimators are decreasing as the correlation coefficient of population is increasing.

(3) In table 1 and 2, the efficiencies of M.M.Es of parameters are increasing as the correlation coefficient of population is increasing.

5. Conclusion and Remark

In this paper, we consider the estimation of parameters for a BVE model of Marshall and Olkin (1967) under the cases of complete and time-truncated parallel sampling. The maximum likelihood estimators have to be found. in general, by complex iterative methods, but the method of moment estimators and Bayes estimators can be derived in closed forms. We knew that whereas existing the M.L.Es of the BVE model have many a good properties, the Bayes estimators under vague prior are better than M.L.Es in small samples.

Table 1. Comparison of the Efficiencies of the Estimators of Parameters for BVE (λ_1, λ_2, λ_0) in the Complete Sampling Case

$(\lambda_1 = \lambda_2 = 1)$

ρ	λ_0	Estimators	n		
			7	10	20
0.048	0.1	M.L.E.	0.7351	0.8126	0.9139
		BAYES E.	0.8977	0.9859	0.9999
		M.M.E.	0.3137	0.3712	0.4994
0.200	0.5	M.L.E.	0.7094	0.7498	0.8824
		BAYES E.	0.8247	0.8165	0.8880
		M.M.E.	0.3969	0.4920	0.6118
0.429	1.5	M.L.E.	0.6070	0.6810	0.8804
		BAYES E.	0.6156	0.6792	0.8750
		M.M.E.	0.4144	0.4834	0.6593

Table 2. Comparison of the Efficiencies of the Estimators of Parameters for BVE (λ_1, λ_2, λ_0) in the Time-Truncated Parallel Sampling Case.

$(\lambda_1 = \lambda_2 = 1,\ x_0 = 0)$

ρ	λ_0	Estimators	n		
			7	10	20
0.048	0.1	M.L.E.	0.5185	0.6152	0.7859
		BAYES E.	0.5673	0.7042	0.8858
		M.M.E.	0.3137	0.3712	0.4994
0.200	0.5	M.L.E.	0.5848	0.6817	0.8274
		BAYES E.	0.6639	0.7412	0.8336
		M.M.E.	0.3969	0.4920	0.6118
0.429	1.5	M.L.E.	0.5444	0.6348	0.8197
		BAYES E.	0.5539	0.6340	0.8164
		M.M.E.	0.4144	0.4834	0.6593

We close this paper with some suggestions in which the present work can be extended:

(1) In this paper, we assume that the distribution is BVE (λ_1, λ_2, λ_0). The estimating procedure for $\underline{\lambda}$ may be extended for the multivariate case.

(2) The M.V.U.E. for $\underline{\lambda}$ in complete or time-truncated parallel case which are not considered in this paper.

(3) The Bayes estimator for λ under the more general loss function which are not considered in this paper.

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