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On a Distribution-Free Test for Parallelism of Regression Lines Against Ordered Alternatives+

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ABSTRACT

A distribution-free rank test for parallelism of regression lines against ordered alternatives is considered. The proposed test statistic is based on the Kepner-Robinson's transformation. The null distribution of the proposed statistic is the same as that of the Wilcoxon signed rank statistic. But, the proposed procedure can be applied only to four or fewer regression lines. The results of a small-sample Monte Carlo study show that the proposed test is comparable with the parametric test in heavy tailed distributions.

1. INTRODUCTION

Consider the linear regression model

$$Y_{ij}=\alpha_i+\beta_i x_{ij}+e_{ij},\quad i=1,\ \cdots,\ k,\quad j=1,\ \cdots,\ N\ .$$

where the x's are known constants, the α's are nuisance parameters, and the β's are regression parameters. The Y's are observable while the e's are mutually independent and identically distributed unobservable random variables with a continuous distribution function.

In this paper we are interested in testing for the parallelism of k regression lines against ordered alternatives, i.e., we want to test

$$H_0:\beta_1=\cdots=\beta_k=\beta\ \text{(unknown)}$$

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against the ordered alternatives

$$H_1 : \beta_1 \leqq \cdots \leqq \beta_k,$$

where at least one inequality is strict. The problem of ordered alternatives could arise in practice. For example, as mentioned in Adichie (1976), a biologist may be interested in knowing whether the rate of dependence of infection on exposure is the same for groups of rats of increasing ages.

In the case of general alternatives, there are many nonparametric tests for the parallelism of regression lines. For example, Hollander (1970) suggested a distribution-free rank test for the parallelism of two regression lines.

However, we are interested in testing for the equality of regression coefficients, i.e. parallelism, against ordered alternatives. Adichie (1976) considered some parametric and nonparametric tests for the parallelism of several regression lines against ordered alternatives. These are the likelihood ratio test and the test based on linear combinations of least squares estimators. Rank analogues of the two tests are also examined. In this paper, we consider a distribution-free rank test and compare the proposed test with a parametric test in terms of empirical powers.

Kepner and Robinson (1984) suggested a distribution-free rank test for ordered alternatives in randomized block designs. Their test statistic is based on a set of transformed variables possessing joint symmetry. In this paper, we want to apply the Kepner-Robinson's transformation for testing the parallelism of four or fewer regression lines. The proposed procedure is an extension of the Hollander's (1970) test. The null distribution of the proposed test statistic is the same as that of the Wilcoxon signed rank statistic. The proposed procedure has some disadvantages: it depends on irrelevant randomization; requires the same number of Y's for each line; and cannot be applied to more than four regression lines. These defects may be considered as a payment for the distribution-free property.

The proposed test is compared with the Adichie's parametric test through a small-sample Monte Carlo study. The results show that the proposed test is not very successful, but it is comparable with the parametric test in heavy tailed distributions.

2. THE TEST PROCEDURES

To introduce the parametric procedure considered by Adichie (1976), we use the following notations.

$$C_i^2 = \sum_{j=1}^{N} (x_{ij} - \bar{x}_i)^2; \quad \bar{x}_i = \sum_{j=1}^{N} x_{ij}/N; \quad \bar{Y}_i = \sum_{j=1}^{N} Y_{ij}/N;$$

$$\hat{\beta}_i = \sum_{j=1}^{N} (x_{ij} - \bar{x}_i) Y_{ij}/C_i^2; \quad \hat{\beta} = \sum_{i=1}^{k} \rho_i \hat{\beta}_i; \quad \rho_i = C_i^2 / \sum_{S=1}^{k} C_S^2 .$$

When k=2, the parametric test statistic for H_0 against H_1 is based on $\hat{\beta}_2 - \hat{\beta}_1$. Extending this idea to the case of k>2, Adichie (1976) suggested the test statistic of the form

$$S = \sum t_i \beta_i, \quad (\sum_{i=1}^{k} t_i = 0). \tag{2.1}$$

The test rejects H_0 for large values of S. Assuming normality, S has a normal distribution with mean

$\sum_{i=1}^{k} t_i\beta_i$ and variance $\sigma^2\sum_{i=1}^{k}(t_i^2/C_i^2)$. If σ^2 is assumed known, the power of S against any given $\beta_1 \leqq \cdots \leqq \beta_k$ is, from Adichie (1976), given by

$$\Phi(\sum t_i\beta_i/\sigma(\sum(t_i^2/C_i^2))^{\frac{1}{2}} - u_\alpha) = \Phi(\gamma\Delta - u_\alpha),$$

where u_α is the upper 100_α % point of the standard normal distribution, Δ^2 is the noncentrality parameter given by $\Delta^2 = \sum C_i^2(\beta_i - \sum\rho_i\beta_i)^2$, and γ is defined by $\gamma = \sum t_i\beta_i/\Delta(\sum(t_i^2/C_i^2))^{\frac{1}{2}}$. Notice that the numerator of γ can be written as

$$\sum t_i\beta_i = \sum C_i(\beta_i - \sum\rho_i\beta_i)(t_i/C_i)$$

Thus, γ represents the correlation coefficient between (t_i/C_i) and $C_i(\beta_i - \sum\rho_i\beta_i)$, $i=1, \cdots, k$. Notice also that the power of S-test for any given β's depends only on γ. Considering this fact, Adichie (1976) used the scores $t_i = C_i^2(S_i - \sum\rho_i S_i)$, where S_i is defined by $S_i = C_i^2 + \cdots + C_{i-1}^2 + C_i^2/2$. With this choice, the statistic S in (2.1) is equivalent to

$$S = \sum_{i<s}^{k} C_i^2 C_s^2(\hat{\beta}_s - \hat{\beta}_i) \ .$$

When σ^2 is not known, S can be studentized to yield S_t which has a Student t distribution with $k(N-2)$ degrees of freedom. The statistic S is defined as

$$S_t = S/(\hat{\sigma}(\sum(t_i^2/C_i^2))^{\frac{1}{2}}) \qquad (2.2)$$

where $\hat{\sigma}^2 = \sum_i\sum_j(Y_{ij} - \overline{Y}_i - \hat{\beta}_i(x_{ij} - \overline{x}_i))^2/(k(N-2))$.

We now consider a distribution-free procedure by using the contrast transformations considered by Kepner and Robinson (1984). Hollander (1970) suggested a distribution-free signed rank test for the parallelism of two regression lines. Actually we consider an extension of the Hollander's procedure to the case of three or four regression lines.

To apply the Hollander's pairing technique we assume that $N=2n$, discarding an observation from each sample if necessary. For each line i, pair the x_{ij}'s to form n groups of the form (x, x'), each containing two unequal x's. For each group compute a slope estimator of β_i of the form $(Y-Y')/(x-x')$ where Y, Y' are the observations corresponding to the paired x, x' points. The grouping cannot depend on the observed Y's but only on the $\{x_{ij}\}$ configurations. Since the Y's for each estimator are different, the n slope estimators are mutually independent for line i. The slope estimators for different lines are also mutually independent. Thus, the slope estimators of the form

$$U_{ij} = (Y_{ij'} - Y_{ij})/(x_{ij'} - x_{ij})$$

are mutually independent for all $i = 1, \ldots, k$ $(k \leqq 4)$ and $j < j' = 1, \cdots, n$.

In order to construct a test statistic for the parallelism of k (2, 3, or 4) regression lines, we consider the following transformations which were originally considered by Kepner and Robinson (1984).

1. When k=2, let $W_{1j}=U_{2j}-U_{1j}$.

2. When k=3, let $W_{1j}=U_{2j}-U_{1j}$; $W_{2j}=2U_{3j}-U_{2j}-U_{1j}$.

3. When k=4, let $W_{1j}=U_{3j}-U_{1j}$; $W_{2j}=U_{4j}-U_{2j}$;
$W_{3j}=U_{4j}-U_{3j}+U_{2j}-U_{1j}$.

We define Ψ_{ij} as the sign of W_{ij}, i.e., Ψ_{ij} = sign (W_{ij}) = 1, 0, or -1 as $W_{ij}>0$, =0, or <0, respectively. We also let R_{ij}^{+} be the rank of $|W_{ij}|$ in the pooled collection of (k-1)n absolute values and define

$$W=\sum_{i=1}^{k-1}\sum_{j=1}^{n}\Psi_{ij}R_{ij}^{+} \quad . \tag{2.3}$$

Then, the null distribution of W is the same as that of the Wilcoxon signed rank statistic. Note that $E_0(W)$ = 0 and W should be close to the expected value under H_0. Thus, we reject $H_0:\beta_1=\cdots=\beta_k$ in favor of $H_1:\beta_1\leqq\cdots\leqq\beta_k$ for significantly large values of W.

The exact critical values of W can be obtained from the tables of exact distribution. The large sample approximation to obtain a critical region is possible by using the fact that $W/[\mathrm{Var}_0(W)]^{\frac{1}{2}}$ has a standard normal limiting distribution under H_0, where

$$\mathrm{Var}_0(W)=n(k-1)[n(k-1)+1][2n(k-1)+1]/6.$$

3. MONTE CARLO RESULTS

In this section we compare the proposed test statistic W in (2.3) with the parametric test statistic S_t in (2.2) by a small-sample Monte Carlo study. We compare the empirical significance levels and empirical powers of the test statistics for various distributions including normal, double exponential, slash and Cauchy distributions.

To generate e_{ij}'s in our simulation study, we used the package IMSL on VAX 780. The uniform random variates are generated by the subroutine GGUBT and the normal random variates are also generated by using the subroutine GGNML. The inverse integral transformation is applied to generate the double exponential and Cauchy random variates.

In our simulation study, the number of regression lines is three (i.e., k=3) and the sample size for each regression line is N = 10. For each i, x_{ij} were fixed with (0.5, 1., ..., 4.5, 5.) and Y_{ij} were obtained from the model

$$Y_{ij}=\alpha_i+\beta_i x_{ij}+e_{ij},\quad i=1,2,3;\ j=1,\ \cdots,\ 10.$$

The β_i's were set at $\beta_1=1$, $\beta_2=\beta_1+m\Delta$, $\beta_3=\beta_1+2m\Delta$, where Δ is fixed at 0.1, 0.1, 0.4 and 0.4 for normal, double exponential, slash and Cauchy distributions, respectively. Chosen values for m were m = 0, 1, 2, 3, or 4. Such setting allows a comparison of the test statistics under the null hypothesis and for increasing equally-spaced alternatives. 500 replications for each experimental situation were performed and the number of times that the null hypothesis H_0 is rejected at nominal 10% level of significance was counted for each set. Then, the number of times of rejecting H_0 divided by 500 is the empirical power. The empirical power at m = 0 is the empirical significance level.

The Monte Carlo results are summarized in Table 1. The results show that the empirical significance levels of the parametric test S_t and the nonparametric test W are very similar. The powers of the parametric test are better than those of the proposed test in medium tailed distribution such as normal and double exponential. But, the proposed test has better powers at heavy tailed distributions, as expected. As a conclusion, the proposed test is not very successful in terms of powers, but it is worthwhile to study on the distribution-free statistic further.

Table 1. Empirical Powers at $\alpha = 0.10$

		$m=0$	$m=1$	$m=2$	$m=3$	$m=4$
Normal	S_t	0.064	0.228	0.268	0.712	0.884
	W	0.074	0.208	0.374	0.610	0.762
Double Exponential	S_t	0.106	0.204	0.354	0.538	0.720
	W	0.092	0.176	0.290	0.432	0.566
Slash	S_t	0.080	0.152	0.240	0.392	0.468
	W	0.084	0.160	0.314	0.434	0.556
Cauchy	S_t	0.118	0.212	0.332	0.486	0.584
	W	0.098	0.224	0.386	0.522	0.650

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