HONAM MATHEMATICAL JOURNAL
Volume 9, Number 1, August 1987

A Note on the Chern Classes***

by

K.A. Lee*, Ho.J. Lee*, He.J. Lee*, D.S. Chun*,
W.K. Jeon*, Y.W. Kim** and I.S. Kim*

* *Dept. of Mathematics, Chonbuk National University, Chonju 520, Korea*
** *Dept. of Mathematics, Hanyang University, Seoul 133, Korea*

Abstract

It is well known that there are two ways to define Chern classes of complex vector bundles. One gives the definition of Chern classes by the five axioms ([2],[3],[4]), and an other defines Chern classes with the associated projective space bundle of a given bundle ([1],[5]).

The purpose of this paper is to describe the latter way in detail and to give new proofs of that our Chern classes satisfy the five axioms with respect to Chern classes (for example Theorem 5).

1. Preliminaries

Every complex vector bundle is orientable. Throughout this paper we shall assume that

(i) every complex vector bundle is oriented by the usual way

(ii) the base space of a given complex vector bundle is Hausdorff and paracompact.

For an n-dimensional complex vector bundle $\xi=(E,\pi,X)$ there exists a unique Euler class

$$e(\xi)\,(\text{or } e(E))\in H^{2n}(X\colon Z) \quad (=H^{2n}(x))$$

of ξ where Z is the integers. The Euler classes setisfy the following properties([2], [3]).

*** The present studies were supported by the Basic Scince Research Institude program, Ministry of Education 1986.
Received May 14, 1987. Accepted June 30, 1987.

Property 1. For two n-dimensional complex vector bundles ξ and ξ', let f: $\xi \longrightarrow \xi'$ be c bundle homomorphism which preserves orientation. Then

$$e(\xi) = g^*(e(\xi'))$$

which is called the naturality of Euler classes.

Property 2. Let ξ and ξ' be complex vector bundles with the same base spase. Then

$$e(\xi \oplus \xi) = e(\xi) \cup e(\xi)' = e(\xi)e(\xi)'$$

where $\cup$ is the Cup product.

For an n-dimensional complex vector bundles $\xi = (E, \pi, X)$

$$\begin{array}{lccc} i: & X & \longrightarrow & E \\ & \Cup & & \cup \\ & x & & 0 \in E_x \end{array}$$

is the zero section $(\pi^{-1}(x) = E_x)$. Putting $E_o = E - \mathrm{i}(X)$, $\xi^- = ((E, E_o), \pi, X, (C^n, C^n - \{0\}))$ is a vector bundle pair. Since

$$H_q(C^n, C^n - \{0\}; Z) = H_q(R^{2n}, R^{2n} - \{0\}) \cong \begin{cases} Z, & q = 2n \\ 0, & g \neq 2n \end{cases}$$

there exists the Thom class $w(\xi) \in H^{2n}(E, E_o : Z)$ and the Thom isomorphism theorem such that

$$\begin{array}{lcl} \phi^*: H^p(X) & \xrightarrow{\cong} & H^{2n+p}(\mathrm{E}, \mathrm{E}_o) \\ \quad \Cup & & \Cup \\ \quad \alpha & \longmapsto & \pi^*(\alpha) \cup w(\xi) = \pi^*(\alpha) w(\xi) \end{array}$$

(Note that π^*: $H^p(X) \xrightarrow{\cong} H^p(E)$ for all $p = 0, 1, \ldots,$). For the inclusion i: $E \hookrightarrow (E, E_o)$, since

$$\begin{array}{ccccc} H^{2n}(E, E_o) & \xrightarrow{i^*} & H^{2n}(E) & \xleftarrow{\pi^*(\cong)} & H^{2n}(X) \\ \Cup & & \Cup & & \\ W(\xi) & \longmapsto & i^*(w(\xi)) & & \pi^{*-1}(i^*(w(\xi))) = e(\xi) \end{array}$$

we have the Thom-Gysin exact suquence:

$$\cdots\cdots \longrightarrow H^{q-1}(E_o) \longrightarrow H^{q-2n}(X) \xrightarrow{\cup e(\xi)} H^q(X) \xrightarrow{\pi_o^*} H^q(E_o) \longrightarrow \cdots\cdots,$$

where $\pi = \pi | E_o$.

we give on ξ the usual Hermitian metric and put

$$Y = \{v \in E \mid \|v\| = 1\},$$

then $H^p(E_o) \cong H^p(Y_o)$ for all $p = 0, 1, 2 \ldots$ Thus the above Thom-Gysin exact sequence becomes as

$$\cdots\cdots \longrightarrow H^{q-1}(Y_o) \longrightarrow H^{q-2n}(X) \xrightarrow{\cup e(\xi)} H^q(X) \longrightarrow H^q(Y_o) \longrightarrow \cdots\cdots \qquad \cdots(*)$$

Proposition 1. Let $P^n(C)$ be the n-dimensional complex projective space, and $P^\infty(C)$ be the infinite complex projective space. Then

$$H^q(P^n(C) : Z) = H^q(P^n(C)) \cong \begin{cases} Z, & q = 0, 2, \ldots, 2n \\ 0, & \text{otherwise} \end{cases}$$

and

$$H^q(P^\infty(C) : Z) = H^q(P^\infty(C)) \cong \begin{cases} Z, & q \text{ is even} \\ 0, & q \text{ is odd} \end{cases}$$

Proof. Let $\pi: L_n \longrightarrow P^n(C)$ be the Hopf line bundle where

$$L_n = \{(V, v) \in P^n(C) \times C^{n+1} \mid v \in V\}$$

and C is the complex. Then there exists the Euler class $e(L_n)$ and $Y_o = S^{2n+1}$. By the Thom-Gysin exact sequence (*) above we have the exact sequence:

$$\begin{aligned} 0 = H^{-1}(P^n(C)) &\to H^1(P^n(C)) \to H^1(L_n) \cong H^1(Y_o) \cong H^1(S^{2n+1}) \to H^0(P^n(C)) \xrightarrow{\cup e(L_n)} \\ & H^2(P^n(C)) \xrightarrow{\pi^*} H^2(S^{2n+1}) \to H^1(P^n(C)) \xrightarrow{\cup e(L_n)} \cdots\cdots \\ \longrightarrow & H^{2n-2}(P^n(C)) \xrightarrow{\cup e(L_n)} H^{2n}(P^n(C)) \xrightarrow{\pi^*} H^{2n}(S^{2n+1}) \longrightarrow H^{2n-1}(P^n(C)) \longrightarrow 0 \\ \longrightarrow & H^{2n+1}(S^{2n+1}) \longrightarrow H^{2n}(P^n(C)) \longrightarrow H^{2n+2}(P^n(C)) = 0, \end{aligned}$$

since $P^n(C)$ is a $2n$-dimensional CW-complex. Noting that

$$H^q(S^{2n+1}) \cong \begin{cases} Z, & q = 2n+1 \text{ or } 0 \\ 0, & q \neq 2n+1 \text{ and } 0 \end{cases}$$

and $H^0(P^n(C)) \cong Z$ we have

$$H^0(P^n(C))\cong\cdots\cdots\cong H^{2n}(P^n(C))\cong \boldsymbol{Z}$$
$$H^1(P^n(C))=H^3(P^n(C))=\cdots\cdots=H^{2n-1}(P^n(C))=0$$

In particular,

$$H^{2i}(P^n(C))=\boldsymbol{Z}(e(L_n))^i \quad (1\leq i\leq n)$$

and thus

$$H^*(P^n(C))=\sum_{i=0}^{\infty}H^i(P^n(C))=\boldsymbol{Z}[1,e]/(e^{n+1}),$$

where (e^{n+1}) is the ideal of $\boldsymbol{Z}[1,e]$ generated by e^{n+1}, and $e=e(L_n)$. Since the inbeddings

$$P^0(C)=\{point\}\hookrightarrow P^1(C)\hookrightarrow\cdots\cdots\hookrightarrow P^n(C)\hookrightarrow\cdots\cdots\hookrightarrow P^{\infty}(C)$$

makes a direct system $\{H^*(P^n(C))\}$ it follows that

$$H^*(P^{\infty}(C))Z[1,e] \quad (e=e(L_n))$$

and

$$H^q(P^{\infty}(C))\cong\begin{cases}Z, & q \text{ is even}\\ 0, & q \text{ is odd.}\end{cases} \quad ///$$

Finally we shall describe the Leray-Hirsch theorem without any proof.

Theorem (L-H) (Leray-Hirsch theorem). Let (E, π, X, F) be a fiber bundle with fiber F. We assume the following:

(i) $H_*(F:Z)=H_*(F)$ is a finite generated Z-free module

(ii) there exists a group homomorphism θ: $H^*(F)\longrightarrow H^*(E)$ (degree zero) such that

$$H^*(F)\xrightarrow{\theta}H^*(E)\xrightarrow{i_x^*}H^*(E_x) \quad \text{(degree zero)}$$

is an isomorphism, where $x\in X$, $\pi^{-1}(x)=E_x$ and $i_x:E_x\hookrightarrow E$ is the inclusion. Then

$$\begin{array}{ccc} H^*(X)\otimes H^*(F) & \cong & H^*(E)\\ \Downarrow & & \Downarrow\\ \alpha\otimes\beta & \longmapsto & \pi^*(\alpha)\cup\theta(\beta)=\pi^*(\alpha)\cdot\theta(\beta)\end{array} \quad ([1],[5]).$$

2. Chern classes

Let $\xi=(E,\pi,X)$ be a n-dimenional complex vector bundle. The associated projective space bundle $\pi'\colon P(E)\longrightarrow X$ is defined as follows. For each $x\in X$ E_x is an n-dimensional complex vector space, and thus the $(n-1)$-dimensional complex projective space $P(E_x)$ is defined. put

$$P(E)=\dot{\bigcup_{x\in X}}P(E_x) \quad (\dot{\bigcup} \text{ means the disjoint union}),$$

then there is the projection $P\colon E\longrightarrow P(E)$. $P(E)$ is topolozied by the quotient topology with respect to P. Then

$$\begin{array}{ccc} \pi'\colon P(E) & \longrightarrow & X \\ \cup & & \Cup \\ P(E_x) & \longmapsto & x \end{array}$$

is a fiber bundle with fiber $P^{n-1}(C)$. Suppose the Hopf line bundle

$$\tilde{\pi}\colon L(E)\longleftarrow P(E)$$

where $L(E)=\{(V,v)\in P(E)\times E\,|\,v\in V\}$ and $\tilde{\pi}((V,v))=V$ (V is a complex line.) Since $\tilde{\pi}\colon L(E)\longrightarrow P(E)$ is a one-dimensional complex vector bundle, there exists the Euler class

$$e(L(E))\in H^2(P(E)).$$

Proposition 2. In the above situation we put $e(L(E))=e$. Then the following hold.

(i) $H^*(P(E))$ $(=\sum_{i=0}^{\infty}H^i(P(E)))$ is a free $H^*(X)$-module generated by $1,e,\dots,e^{n-1}$.

(ii) For each $\omega\in H^*(P(E))$ there exists a unique reprasentation of w such that

$$w=\sum_{i=0}^{n-1}\pi'^*(v_i)e^i,$$

where $v_i\in H^*(X)$.

Proof. (i) Recall the fiber bundle $\pi'\colon P(E)\longrightarrow X$ with fiber $P(E_x)\cong P^{n-1}(C)$ for each $x\in X$. For the inclusion $i_x\colon P(E_x)\longrightarrow P(E)$ we put

$$i_x^*(e)=e_x\in H^2(P(E_x)),$$

then by property 1° e_x is the Euler class of the Hopf-line bundle over $P(E_x)$ ($\simeq p^{n-1}(C)$). By proposition 1

$$H^*(P(E_x))=Z[1,e_x]/(e_x^n).$$

(Note that $i_x^*(e^i)=e_x^i$ by the property of cup products.) Define

$$\begin{array}{cccc} \theta: & H^*(P(E_x)) & \longrightarrow & H^2(P(E)) \\ & \Cup & & \Cup \\ & e_x^i & \longmapsto & e^i \end{array}$$

then θ is a ring homomorphism and $i_x^*\theta=1_{H^*(P(E_x))}$. Thus by Theorem($L$-$H$) we have the isomorphism

$$\begin{array}{ccc} H^*(X)\otimes H^*(P^{n-1}(C)) & \cong & H^*(P(E)) \\ \Cup & & \Cup \\ v_i\otimes(e')^j & \longmapsto & \pi'^*(v_o)\cup e^i=\pi'^*(v_i)e^i \end{array}$$

where e' is the Euler class of the Hopf line bundle over $p^{n-1}(C)$. Since the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \pi\ } & E \\ & \nwarrow^{\pi'} \quad © \quad \swarrow^{p} & \\ & P(E) & \end{array}$$

induces the commutative diagram

$$\begin{array}{ccc} H^*(X) & \xrightarrow[\cong]{\ \pi^*\ } & H^*(E) \\ & \nwarrow^{\pi'^*} \quad © \quad \swarrow^{p^*} & \\ & H^*(P(E)) & \end{array}$$

it follows that π'^* is a monomorphism and p^* is an epimorphism. Therefore, the cohomology ring $H^*(P(E))$ is a free $H^*(X)$-module generated by $\{1,e,\dots,e^{n-1}\}$.

(ii) Since π'^* is injective it is clear that (ii) is true. ///

Definition 3. In the above situation it is obvious that there is a unique subset

$\{C_1(\xi),\dots,C_n(\xi)\}\subset H^*(X)$ $(G(\xi)\in H^{2i}(X)$ satisfying

$$\sum_{i=1}^{n-1}\pi'^*(C_{n-i}(\xi))(-e)^i+(-e)^n=0 \quad \text{(by proposition 2)}.$$

(Sometimes we put $C_i(\xi)=C_i(E)$.) We shall put

$$C_0(\xi)=1 \quad C_i(\xi)=0 \text{ if } i>n,$$

and call $\{C_0(\xi),\ C_1(\xi),\dots,C_n(\xi)\}$ $(\subset H^*(X))$ the Chern classes of ξ (Note that since $H^0(X)\cong Z$, $1\in H^0(X)$).

Lemma 4. (i) Let $\xi=(E,\pi,p^1(C))$ be the Hopf line bundle over $p^1(C)$. Then $C_1(\xi)$ is the generator of $H^2(p^1(C))$.

(ii) Let $\xi=(E(\xi),\pi_\xi,X(\xi))$ and $\eta=(E(\eta),\pi_\eta,X(\eta))$ be n-dimensional complex vector bundles, and $(\bar f, f)$: $\xi\longrightarrow\eta$ be a bundle map such that

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\ \bar f\ } & E(\eta) \\ \pi_\xi\downarrow & & \downarrow\pi_\eta \\ X(\xi) & \xrightarrow{\ f\ } & X(\eta). \end{array}$$

Then $C_i(\xi)=f^*(C_i(\eta))$ for $i=0,1,\dots,n$.

Proof. (i) since the associated complex projective space bundle $P(E)$ of ξ is equal to $p^1(C)$. by Definition 3

$$e(\xi)=C_1(\xi).$$

By Proposition 2 $e(\xi)$ is the generator of $H^2(p^1(C))$. Therefore $C_1(\xi)$ is the generator of $H^2(p^1(C))\cong Z$

(ii) It is clear that

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\ \bar f\ } & E(\eta) \\ \pi_\xi\Big\downarrow & © & \Big\downarrow\pi_\eta \\ X(\xi) & \xrightarrow{\ f\ } & X(\eta) \end{array} \quad\Longrightarrow\quad \begin{array}{ccc} P(E(\xi)) & \xrightarrow{\ \bar f'\ } & P(E(\eta)) \\ \pi'_\xi\Big\downarrow & © & \Big\downarrow\pi_\eta' \\ X(\xi) & \xrightarrow{\ f\ } & X(\eta), \end{array}$$

where $P(E(\xi))$ and $P(E(\eta))$ are the associated complex projective space bundles of ξ and η, respectively, and for each $x\in X(\xi)$, for each $V_x\in P(E(\xi))$ $\bar f'(V_x)=\bar f(v_x)$

$(=V_{f(x)}\in P(E(\eta)))$. Let $\tilde{\pi}_\xi: L(E(\xi))\longrightarrow P(E(\xi))$ and $\tilde{\pi}_\eta: L(E(\eta))\longrightarrow P(E(\eta))$ be the Hopt line bundles over $P(E(\xi))$ and $P(E(\eta))$, respectively. Then we have the commutative diagram

$$\begin{array}{ccc} L(E(\xi)) & \xrightarrow{\tilde{f}} & L(E(\eta)) \\ \tilde{\pi}_\xi \downarrow & f' & \downarrow \tilde{\pi}_\eta \\ P(E(\xi)) & \longrightarrow & P(E(\eta)), \end{array}$$

where for each $(V_x, v)\in L(E(\xi))$, $\tilde{f}(V_x, v)=(f(V_x), f(v))$. Let $e(\xi)\in H^2(P(E(\xi))$ and $e(\eta)\in H^2(P(E(\eta)))$ be the Euler classes of $L(E(\xi))$ and $L(E(\eta))$, respectively. By property 1°

$$f'^*(e(\eta))=e(\xi).$$

For the Euler classes$\{1, C_1(\eta), \ldots, C_n(\eta)\}$ of η we have

$$\sum_{i=0}^{n-1}\pi_\eta'^*(C_{n-i}(\eta))(-e(\eta))^i+(-e(\eta))^n=0$$

Thus

$$0=f'^*(\sum_{i=0}^{n-1}(\pi_\eta'^*(C_{n-i}(\eta))(-e(\eta))^i+(-e(\eta))^\eta$$

$$=\sum_{i=0}^{n-1}f'^*\pi_\eta'^*(C_{n-i}(\eta))(-(f)^*(e(\eta)))^i+(-f^*(e(\eta)))^n$$

$$=\sum_{i=0}^{n-1}\pi_\xi'^*(f^*(C_{n-i}(\eta))(-e(\xi))^i+(-e(\xi))^n$$

(Note that $f'^*\pi_\eta'^*=\pi_\xi'^*\circ f^*$). By (ii) of Proposition 2

$$f^*(C_{n-i}(\eta))=C_{n-i}(\xi)$$

for $i=0, 1, \ldots, n-1$. ///

Theorem 5. Let $\xi=(E(\xi), \pi_\xi, X)$ and $\eta=(E(\eta), \pi_\eta, X)$ be complex vector bundles over the same base space X. Then

$$C_k(\xi\oplus\eta)=\sum_{i+j=k}C_i(\xi)\cup C_j(\eta)=\sum_{i+j=k}C_i(\xi)C_j(\eta).$$

Proof. We assume that dim ${}_c\xi=P$ and dim ${}_c\eta=q$. Suppose the associated complex projective space bundles

$$P(E(\xi)\oplus E(\eta)),\ P(E(\xi)) \text{ and } P(E(\eta))$$

of $\xi\oplus\eta$, ξ and η, respectively. Then for each $x\in X$ it follows that

$$P(E_x(\xi)\oplus E_x(\eta))\approx P^{p+q}(C),\ P(E_x(\xi))\approx P^p(C),\ P(E_x(\eta))\approx P^q(C),$$

where $\approx$ means to be homeomorphic and $E_x(\xi)=\pi_\xi^{-1}(X)$.

Define inclusions

$$\begin{array}{ccc} i_{x,\xi}\colon P(Ex(\xi)) & \hookrightarrow & P(E_x(\xi)\oplus E_x(\eta)) \\ \Downarrow & & \Downarrow \\ [v_1,\dots,v_p] & \longmapsto & [v_1,\dots,v_p,0,\dots,0] \end{array}$$

and

$$\begin{array}{ccc} i_{x,\eta}\colon P(E_x(\eta)) & \hookrightarrow & P(E_x(\xi)\oplus E_x(\eta)) \\ \Downarrow & & \Downarrow \\ [v_{p+1},\dots,v_{p+q}] & \longmapsto & [0,\dots,0,v_{p+1},\dots,v_{p+q}] \end{array}$$

where $(v_1,\dots,v_p)\in E_x(\xi)\cong C^p$ and $[v_1,\dots,v_p]\in P(E_x(\xi))\cong P^{p-1}(C)$. It follows that

$$P(E_x(\xi)\oplus E_x(\eta))-P(E_x(\xi))=\{[v_1,\dots,v_{p+q}]\in P(E_x(\xi)\oplus E_x(\eta))\mid \ni \text{ at least one } x_i\neq 0 \text{ for } p+1\leq i\leq p+q\}$$

Define

$$\begin{array}{ccc} \gamma_x(\eta)\colon P(E_x(\xi)\oplus E_x(\eta))-P(E_x(\xi)) & \longrightarrow & P(E_x(\eta)) \\ \Downarrow & & \Downarrow \\ [v_1,\dots,v_{p+q}] & \longmapsto & [v_{q+1},\dots,v_{p+q}]\ (\neq 0) \end{array}$$

then it follows that $\gamma_x(\eta)$ is a deformation retraction of $P(E_x(\xi)\oplus E_x(\eta))-P(E_x(\xi))$. Hence there exist inclusions

$$i_\xi\colon P(E(\xi))\longrightarrow P(E(\xi)\oplus E(\eta)),\ i_\eta\colon P(E(\eta))\longrightarrow P(E(\xi)\oplus E(\eta))$$

and deformation retractions

$$\gamma(\xi)\colon P(E(\xi)\oplus E(\eta))-P(E(\eta))\longrightarrow P(E(\xi))$$
$$\gamma(\eta)\colon P(E(\xi)\oplus E(\eta))-P(E(\xi))\longrightarrow P(E(\eta)).$$

Note that

$$\gamma(\xi)^*:\ H^*(P(E(\xi)))\cong H^*(P(E(\xi)\oplus E(\eta))-P(E(\eta)))$$
$$\gamma(\eta)^*:\ H^*(P(E(\eta)))\cong H^*(P(E(\xi)\oplus E(\eta))-P(E(\xi))).$$

Suppose the commutative diagrams:

$$\begin{array}{ccc} P(E(\xi)) & \xrightarrow{i_\xi} & P(E(\xi)\oplus E(\eta)) \\ & {\scriptstyle \pi_\xi'}\searrow \quad \swarrow{\scriptstyle \pi_{\xi+\eta}} & \\ & X & \end{array}$$

and

$$\begin{array}{ccc} P(E(\xi)\oplus E(\eta))-P(E(\eta)) & \xrightarrow{\gamma(\xi)} & P(E(\xi)) \\ & {\scriptstyle \pi''}\searrow \quad \swarrow{\scriptstyle \pi_\xi'} & \\ & X & \end{array}$$

Let $e\in H^2(P(E(\xi)\oplus E(\eta)))$ be the Euler class of the Hopf line bundle over $P(E(\xi)\oplus E(\eta))$. Then by property 1°

$$i^*(e)\in H^2(P(E(\xi)))$$

in the Euler class of the Hopf line bundle over $P(E(\xi))$. In this case

$$\gamma(\xi)^*i^*(e)\in H^2(P(E(\xi)\oplus E(\eta))-P(E(\eta))\cong H^2(P(E(\xi))).$$

Let

$$\{1,\ C_1(\xi),\dots,C_p(\xi)\}$$

be the Chern classes of ξ. Then, by Definition 3

$$\sum_{i=0}^{p}\pi_\xi'^*(C_{p-i}(\xi))(-i_\xi^**(e))^i=0$$

Therefore

$$\gamma(\xi)*(\sum_{i=0}^{p}\pi_\xi'^*(C_{p-i}(\xi))(-i_\xi^*(e))^i)$$
$$=\sum_{i=0}^{p}\pi''^*(C_{p-i}(\xi))(-\gamma(\xi)^*i_\xi^*(e))^i=0$$

Since

$$\pi'' = \pi'_{\xi+\eta}|_{P(E(\xi)\oplus E(\eta))} - P(E(\eta))$$
$$\gamma(\xi)*i*(e) = e|P(E(\xi)\oplus E(\eta)) - P(E(\eta)),$$

the above identity can be denoted by

$$\sum_{i=0}^{p}\pi'^{*}_{\xi\oplus\eta}(C_{p-i}(\xi))(-e)^i|P(E(\xi)) - P(E(\eta)) = 0$$

This means that

$$\sum_{i=0}^{p}\pi'^{*}_{\xi\oplus\eta}(C_{p-i}(\xi))(-e)^i \in H^*(P(E(\xi)\oplus E(\eta)),\ P(E(\xi)\oplus E(\eta)) - P(E(\eta))).$$

Similarly, for the Chern classes

$$\{1^{-}, C_1(\eta), \ldots, C_q(\eta)\}$$

of η w have

$$\sum_{j=0}^{q}\pi'^{*}_{\xi\oplus\eta}(C_{q-j}(\eta))(-e)^i|P(E(\xi)\oplus E(\eta)) - P(E(\xi)) = 0$$

and

$$\sum_{i=0}^{q}\pi'^{*}_{\xi\oplus\eta}(C_{q-j}(\eta))(-e)^i \in H^*(P(E(\xi)\oplus E(\eta)),\ P(E(\xi)\oplus E(\eta)) - P(E(\xi))).$$

since $P(E(\eta))$ is a deformation retract of $P(E(\xi)\oplus E(\eta)) - P(E(\xi))$ $(P(E(\xi)\oplus E(\eta)) - P(E(\eta)))\cup(P(E(\xi)\oplus E(\eta)) - P(E(\xi))$ is a deformation retract of $P(E(\xi)\oplus E(\eta))$. Therefore we have

$$H^*(P(E(\xi)\oplus E(\eta)),\ P(E(\xi)\oplus E(\eta)) - P(E(\eta)))\cup(P(E(\xi)\oplus E(\eta)) - P(E(\xi))) \cong H^*(P(E(\xi)\oplus E(\eta),\ P(E(\xi)\oplus E(\eta))) = 0$$

Hence

$$0 = \sum_{i=0}^{p}\pi'^{*}_{\xi\oplus\eta}(C_{p-i}(\xi))(-e)^i \cup \sum_{j=0}^{q}\pi'_{\xi\oplus\eta}(C_{q-j}(\eta))(-e)^j$$
$$= \sum_{k=0}^{n}\pi'^{*}_{\xi\oplus\eta}\left(\sum_{i+j=k}(C_{p-i}(\xi)\cup C_{q-j}(\eta))(-e)^k\right.$$

and we have

$$C_k(\xi\oplus\eta) = \sum_{i+j=k}C_i(\xi)\cup C_j(\eta) = \sum_{i+j=k}C_i(\xi)C_j(\eta). \quad ///$$

Lemma 6. Let $\xi=(E,\pi,X)$ be an n-dimensional complex vector bundle. Then

$$C_n(\xi)=e(\xi) \quad \text{(the Euler class of } \xi).$$

Proof. At first, we assume that $\xi=(E,\pi,X)$ is a one-dimensional complex vector bundle. For the associated complex projective space bundle $P(E)$ and the Hopf line bundle $\tilde{\pi}$: $L\to P(E)$ we have a bundle homomorphism $(\tilde{\pi},\pi')$: $L\longrightarrow\xi$ such that

$$\begin{array}{ccc} L & \xrightarrow{\tilde{\pi}'} & E \\ \tilde{\pi}\downarrow & & \downarrow\pi \\ P(E) & \xrightarrow{\pi'} & X \end{array}$$

Let $e(E)\in H^2(X)$ be the Euler class of ξ. Then

$$\pi'^*(e(E))=e(L)\in H^2(P(E))$$

is the Eular class of $L(\tilde{\pi}$: $L\longrightarrow P(E))$. By Definition 3

$$\pi'^*(C_1(E))=e(L)=\pi'^*(e(E)),$$

where $\{1,C_1(E)\}$ is the Euler classes of ξ. Since π'^* is injective(see in the proof of Proposition 2) we have

$$C_1(E)=e(E).$$

In general case the base space X is Hausdorff and paracompact by our assumption. Thus, there exists a classifying map g: $X_1\longrightarrow X$ satisfying

(i) $g^*\xi$ is a Whitney sum of complex line bundles

(ii) g^*: $H^*(X)\longrightarrow H^*(X_1)$ is injective

([4]) Thus

$$g^*\xi=\xi_1^1\oplus\ldots\oplus\xi_n^1$$

where ξ_i^1 $(1\leq i\leq n)$ is a complex line bundle over X_1. By (ii) of Lemma 4

$$\begin{aligned} g^*(C_n(\xi))&=C_n(g^*(\xi))=C_n(\xi_1^1\oplus\ldots\oplus\xi_n^1)\\ &=C_1(\xi_1^1)\ldots C_1(\xi_n^1) \quad \text{(by Theorem 5).}\end{aligned}$$

On the other hand

$$\begin{aligned} g^*(e(\xi)) &= e(g^*(\xi)) \quad \text{(by Property 1°)} \\ &= e(\xi_1{}^1 \oplus \ldots \oplus \xi_n{}^1) \\ &= e(\xi_1{}^1) \ldots e(\xi_n{}^1) \quad \text{(by property 2°)} \\ &= C_1(\xi_1{}^1) \ldots C_1(\xi_n{}^1) \quad \text{(by the first part of this proof).} \end{aligned}$$

Thus, $g^*(C_n(\xi)) = g^*(e(\xi))$ and since g^* is injective we have $C_n(\xi) = e(\xi)$. ///

In consequence we proved that our Chern classes satisfy the five axioms for Chern classes by Proposition 2, Lemma 4. Theorem 5 and Lemma 6.

References

1. A. Hattori: *Topology Vol.* Ⅲ. Iwanami Book Company (1979).
2. K. A. Lee: *Fundations of Topology Vol* Ⅱ. Hakmunsa (1983).
3. J.W. Milnor and J.D. Stashff: *Characteristic Classes*, Princeton University Press (1974).
4. M. Nakaoka: *Topology*, Kyoritzsha (1970).
5. I. Tamura: *Differentiable Topology*, Iwanami Book Company (1979).