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Asymptotic Efficiency of a Nonparametric Test in Nonlinear Regression Parameters

by

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1. Introduction

A Problem that is often encountered in both the linear and nonlinear regression analysis is the nonnormality of errors in the model. The classical method of testing hypotheses about regression parameters assumes that the errors (so observations) are normally distributed. Under this assumption, the likelihood ratio criterion provides the most powerful test. Quite often, however, the normality assumption of the errors is not appropriate. Rather there is good reason to admit the non-normality. When this is the case, the likelihood ratio test statistic calculated on the wrong assumption of nomality fails to perform.

In such a situation, the distribution free methods, especially the rank method, may provide a good answer to the testing problem. This is so because in many rank methods, it is enough to assume that the distribution function of the observations is continuous. In fact, the chief merit of nonparametric tests lies in their generality, and it seems useful that an assessment of their performance is not restricted by the intrinsic postulates in their parametric tests.

This study is concerned with the test of the hypothesis:

$$H_0:\ \theta=\theta_0. \qquad \cdots\cdots (1.1)$$

When the observations are responses Y_t to input x_t generated according to the nonlinear regression model

$$Y_t=f(x_t,\ \theta)+\varepsilon_t,\ \ t=1,2,\cdots,n \qquad \cdots\cdots (1.2)$$

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The functional form of the response function $f(x_t, \theta)$ is known function, θ is the unknown $p\times 1$ parameter vector from a compact parameter space $\Theta\subset R^p$. The errors ε_t are assumed to be independent and identically distributed with zero mean, finite variance and with the common absolutely continuous distribution G which has absolutely continuous density g, with finite Fisher information, i.e.,

$$I(g)=\int_{-\infty}^{\infty}(g'/g)^2dG<\infty.$$

The problem of the testing hypotheses about the unknown parameter θ appearing in a nonlinear model has received almost no attention in the statistical literature. However, in the case of the linear regression model $f(x_t, \theta)=x_t'\theta$, where x_t is a $p\times 1$ vector, the use of rank tests as alternative to classical tests has been discussed by many authors: Pitman (11), Hodges and Lehmann (7), Chernoff and Savage (3), Hajek (5), Puri and Sen (12) among them.

Whether the given regression model is linear or nonlinear it is reasonable to evaluate the performance of any testing procedure. The usual measure used is the power of test against alternative hypothesis. Power calculations require knowledge of the probability distribution of the test statistic under the alternative hypothesis, but the derivations of their even asymptotic distributions may often be impossible. Furthermore, a test is consistent against any fixed alternative when the power of the test asymptotically tends to 1 as the sample size increase. Hence, for the study of the asymptotic power properties, one confines oneself to some local alternatives for which the power approaches to an arbitrary number less than 1. Such local alternatives tend to the null hypothesis as the sample sizes increase, and Hájek (5) has studied the asymptotic normality of the test statistic under such local alternatives using the notion of contiguity, which is due to LeCam (9).

In this study we shall employ the sequence of contiguous alternative

$$H_1:\ \theta=\theta_{(n)}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(1.3)$$

where $\lim_{n\to\infty}\theta_{(n)}=\theta_0$, and use the concept of asymptotic efficiency as defined by Pitman (See (11))

The contents of the study are as follows.

In section 2, a class of signed rank order statistic is proposed and the asymptotic distribution of the statistic is shown to be normal under the null hypothesis and a certain regularity conditions about regression function. In Section 3 we introduce suitable families of contiguous alternatives and derive the asymptotic normality of signed rank order statistic under such alternatives. Also, we define a quadratic form

$$Q = S' \Sigma_n^{-1} S \cdots\cdots (1.4)$$

as the test statistics, and, hence, we establish that the test statistics have a asymptotic central chi-square distribution under the null hypothesis and a asymptotic noncentral chi-square distribution under the contiguous sequence of alternatives. Here S is a $p \times 1$ random vector and Σ_n a $p \times p$ covariance matrix of the signed rank order statistic, respectively. Finally, in Section 4, the asymptotic relative efficiency of the proposed statistics Q relative to the least squares test statistic will be studied by using the ratio of their noncentrality parameters.

2. Asymptotic Distribution Under Null Hypothesis

2.1 Asymptotic Normality of S_j under H_o.

We consider random variables $Y_t, t=1, \cdots n$, which are considered as observable and satisfy the equations (1.2). As mentioned before, errors ε_t in (1.2) are independent random variables with the absolutely continuous distribution function G.

In this section, we construct a signed rank order statistic, and then derive the asymptotic normality of the statistic under the null hypothesis. For this we need the following assumptions in the sequel.

Assumption A. The distribution function G satisfy

(i) $g(y) = dG(y)/dy$ exists and is absolutely continuous on R.

(ii) $\phi(u) = -(g'/g)\ (G^{-1}(u+1)/2))$ can be written as the sum of two square integrable functions $\phi_1(u)$ and $\phi_2(u)$, where $\phi_1(u)$ is non-decreasing and non-negative and $\phi_2(u)$ is non-increasing and non-positive on $0<u<1$,

(iii) g is symmetric, that is $g(y) = g(-y)$ for all y.

If G is the logistic distribution function $G(y) = \{1+e^{-y}\}^{-1}$, we easily see that $\phi(u) = u$. Also, if G is chosen to be the normal distribution function ϕ, then $\phi(u) = \phi^{-1}((u+1)/2)$.

We also need some assumptions of the regression function $f(x, \theta)$.

Assumption B. All the first order partial derivatives, $\dfrac{\partial f(x_t, \theta)}{\partial \theta_j}$, $j=1, \cdots, p$, exist and satisfy the following conditions;

$$\text{(i)}\quad \frac{\partial f(x_t,\theta)}{\partial \theta_j} \text{ is a continuous on } \Theta,$$

$$\text{(ii)}\quad \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \left[\frac{\partial f(x_t,\theta)}{\partial \theta_j} \right]^2 < \infty.$$

Let R_t be the rank of $|Y_t - f(x_t,\theta)|$ in the sequence of absolute values $|Y_1 - f(x_1,\theta)|, \cdots, |Y_n - f(x_n,\theta)|$ of the n quantities. Consider the rank order statistic S_j defined by

$$S_j = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \frac{\partial f(x_t,\theta)}{\partial \theta_j} \phi\left(\frac{R_t}{n+1}\right) \mathrm{Sign}(Y_t - f(x_t,\theta))$$

for $j=1,\cdots,p$ where Sign $(v)=1, 0, -1$ as $v>0$, $=0$, <0, respectively.

The following lemma gives the asymptotic null distribution of S_j for $j=1,\cdots,p$, under Assumptions A and B.

Lemma 2.1. Let Assumptions A and B be satisfied. Under the null hypothesis, $S_j(j=1,\cdots,p)$ is asymptotically normal distribution with

$$E(S_j)=0$$

and

$$\mathrm{Var}(S_j) = \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \left[\frac{\partial f(x_t,\theta)}{\partial \theta_j} \right]^2 \int_0^1 \phi^2(u)\,du$$

Proof. Now introduce statistics

$$L_j = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \frac{\partial f(x_t,\theta)}{\partial \theta_j} \phi(G^*(|Y_t - f(x_t,\theta)|))\ \mathrm{Sign}(Y_t - f(x_t,\theta))$$

where G^* is the distribution function of $|Y_t - f(x_t,\theta)|$, i.e., $P_0\{|Y_t - f(x_t,\theta)| \le e\} = G^*(e)$ for $e>0$. Note that if we let

$$Z_{tj} = \frac{1}{\sqrt{n}} \frac{\partial f(x_t,\theta)}{\partial \theta_j} \phi(U_t)\ \mathrm{Sign}(Y_t - f(x_t,\theta))$$

where $U_t = G^*(|Y_t - f(x_t,\theta)|)$ which are independent variables uniformly distributed on (0.1), then $L_j = \sum_{t=1}^{n} Z_{tj}$ with variance $\sigma_L{}^2$;

$$\sigma^2_{L_j} = \frac{1}{n} \sum_{t=1}^{n} \left[\frac{\partial f(x_t,\theta)}{\partial \theta_j} \right]^2 \int_0^1 \phi^2(u)\,du$$

For the independent identically distributed random variables $\phi(U_t)$, $t=1,\cdots,n$, the $2n$ random variables

$$\phi(U_1),\cdots,\phi(U_n),\ \mathrm{Sign}(Y_1-f(x_t,\theta)),\cdots,\ \mathrm{Sign}(Y_n-f(x_t,\theta))$$

are mutually independent. Then, the Lindeberg condition of the central limit theorem amounts to showing

$$\lim_{n\to\infty}\frac{1}{\sigma^2_{L_j}}\sum_{t=1}^{n}\int_{|z_t|>\varepsilon\sigma_{L_j}} z^2_{t,j}\,dF(Z_{t,j})=0\cdots\cdots\cdots\cdots\cdots\cdots(2.1)$$

for every $\varepsilon>0$. Before taking the limit, the left side of (2.1) is

$$\frac{1}{\sigma^2_{L_j}}\sum_{t=1}^{n}\Big[\int_{\{u:\frac{1}{\sqrt{n}}|\frac{\partial f(x_t,\theta)}{\partial\theta_j}\phi(u)|>\varepsilon\sigma_{L_j}\}}\Big(\frac{1}{\sqrt{n}}\frac{\partial f(x_t,\theta)}{\partial\theta_j}\phi(u)\Big)^2du\Big]$$
$$\leq\Big[\int_0^1\phi^2(u)\,du\Big]^{-1}\int_{\{u:\,|\phi(u)|>\varepsilon\delta_n\}}\phi^2(u)\,du\Big]\cdots\cdots\cdots(2.2)$$

where

$$\delta_n=\Big[\int_0^1\phi^2(u)\,du\Big]^{\frac{1}{2}}\frac{\Big\{\frac{1}{n}\sum_{t=1}^{n}\Big[\frac{\partial f(x_t,\theta)}{\partial\theta_j}\Big]^2\Big\}^{\frac{1}{2}}}{\max_{1\leq t\leq n}\frac{1}{\sqrt{n}}\Big|\frac{\partial f(x_t,\theta)}{\partial\theta_j}\Big|}$$

Furthermore, Assumption B implies that $\delta_n\to\infty$ as $n\to\infty$, and then the upper bound (2.2) goes to zero. Therefore, (2.1) is satisfied and L_j is asymptotically normally distributed.

Because the vactors $(R_1,\cdots,R_n)$, $(|Y_1-f(x_1,\theta)|,\cdots,|Y_n-f(x_n,\theta)|)$ and $(\mathrm{Sign}(Y_1-f(x_1,\theta)),\cdots,\mathrm{Sign}(Y_n-f(x_n,\theta)))$ are mutually independent, and $E(\mathrm{Sign}(Y_t-f(x_t,\theta_0)))=0$ for all t, we obtain

$$E_0[(S_j-L_j)^2]=\frac{1}{n}\sum_{t=1}^{n}\Big[\frac{\partial f(x_t,\theta)}{\partial\theta_j}\Big]^2\,\mathrm{E}_0\Big[\phi\Big(\frac{R_t}{n+1}\Big)-\phi(U_t)\Big]^2.$$

Clearly,

$$E\Big[\phi\Big(\frac{R_t}{n+1}\Big)-\phi(U_t)\Big]^2=E\Big[\phi\Big(\frac{R_1}{n+1}\Big)-\phi(U_1)\Big]^2$$

for every $t=1,\cdots,n$. Using Theorem 8.3.23 of (13) it is seen that

$$S_j - L_j \longrightarrow 0 \text{ in probability, } j=1,\cdots,p.$$

This completes the proof. ///

3. Asymptotic Distributicn Under Contiguous Alternatives

3.1 Asymptotic Normality of S_j Under H_1.

The limiting distributions of test statistics under alternative plays a vital role in the study of the asymptotic power. Particularly, for the evaluation of the efficiency of the consistent test statistics, it is necessary to find its distribution function under a sequence of alternatives tending to the null hypothesis at a suitable rate. In this section we consider the distribution of S_j for contiguous alternative tending to H_0 at a certain rate. For this, we shall follow the method based on LeCam's contiguity lemma (6).

Let Q_n be the distribution function of $\{Y_1 - f(x_1, \underset{\sim}{\theta}), \cdots, Y_n - f(x_n, \underset{\sim}{\theta})\}$ under the alternative hypotheses H_1 and let P_0 be the corresponding distribution function under the null hypothesis H_0. If q_n and p_0 are densities associated with the distribution functions Q_n and P_0, respectively, we have

$$q_n = \prod_{t=1}^{n} g(Y_t - f(x_t, \underset{\sim}{\theta}_{(n)}))$$

and

$$P_0 = \prod_{t=1}^{n} g(Y_t - f(x_t, \underset{\sim}{\theta}_0))$$

We proceed to verify the asymptotic normality of W_n where

$$W_n = 2\sum_{t=1}^{n}\left[\left\{\frac{g(Y_t - f(x_t, \underset{\sim}{\theta}_{(n)}))}{g(Y_t - f(X_t, \underset{\sim}{\theta}_0))}\right\}^{\frac{1}{2}} - 1\right].$$

Toward this, let

$$S_n^* = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} [f(x_t, \underset{\sim}{\theta}_0) - f(x_t, \underset{\sim}{\theta}_{(n)})] \left\{-\frac{g'(Y_t - f(x_t, \underset{\sim}{\theta}_0))}{g(Y_t - f(x_t, \underset{\sim}{\theta}_0))}\right\} \cdots\cdots(3.1)$$

and

$$S(Y_t - f(x_t, \underset{\sim}{\theta})) = [g(Y_t - f(x_t, \underset{\sim}{\theta}))]^{\frac{1}{2}}$$

so that

$$S'(Y_t - f(x_t, \underset{\sim}{\theta})) = \frac{1}{2}\left[\frac{\{g'(Y_t - f(x_t, \underset{\sim}{\theta}))\}^2}{g(Y_t - f(x_t, \underset{\sim}{\theta})}\right]^{\frac{1}{2}}.$$

Also, we need the following assumptions in the suquel.

Assumption C. The function $f(x_t, \underset{\sim}{\theta})$ is continuous on Θ and satisfies the following conditions;

$$0 < \lim_{n\to\infty} \sum_{t=1}^{n} [f(x_t, \theta_0) - f(x_t, \underset{\sim}{\theta}_{(n)})]^2 < \infty$$

where $\lim_{n\to\infty} \underset{\sim}{\theta}_{(n)} = \theta_0$.

We are now ready to state the asymptotic normality of $\sqrt{n}\, S_n^*$

Lemma 3.1. Let Assumption A and C be satisfied. Under the alternative hypothesis, $\sqrt{n}\, S_n^*$ is asymptotically normally distributed with

$$E(\sqrt{n}\, S_n^*) = 0$$

and

$$\text{var}\ (\sqrt{n}\, S^*) = \lim_{n\to\infty} \sum_{t=1}^{n} [f(x_t, \underset{\sim}{\theta}_0) - f(x_t, \underset{\sim}{\theta}_{(n)})]^2 \int_0^1 \psi^2(u)du$$

where g^* is the density function corresponding to the distribution function G^* and $\psi(u) = -g^{*\prime}(|Y_t - f(x_t, \underset{\sim}{\theta}_0)|)/g^*(|Y_t - f(x_t, \underset{\sim}{\theta}_0)|)$.

Proof. Let

$$Z_n^* = [f(x_t, \underset{\sim}{\theta}_0) - f(x_t, \underset{\sim}{\theta}_{(n)})]\left\{-\frac{g'(Y_t - f(x_t, \underset{\sim}{\theta}_0))}{g(Y_t - f(x_t, \underset{\sim}{\theta}_0))}\right\}.$$

Because of the symmetry of the density function g, we may write

$$\begin{aligned} Z_n^* &= [f(x_t, \underset{\sim}{\theta}_0) - f(x_t, \underset{\sim}{\theta}_{(n)})]\left\{-\frac{g^{*\prime}(|Y_t - f)(x_t, \underset{\sim}{\theta}_0)|)}{g^*(|Y_t - f(x_t, \underset{\sim}{\theta}_0)|)}\right\} \times \text{Sign}\ (Y_t - f(x_t, \underset{\sim}{\theta}_0)) \\ &= [f(x_t, \underset{\sim}{\theta}_0) - f(x_t, \underset{\sim}{\theta}_{(n)})]\ \psi(U_t)\ \text{Sign}(Y_t - f(x_t, \theta_0)) \end{aligned}$$

It can be easily seen that Z_n^* are independently distributed random variables having expected value zero and variance,

$$\text{Var}(Z_n^*) = [f(x_t, \underset{\sim}{\theta}_0) - f(x_t, \underset{\sim}{\theta}_{(n)})]^2 \int_0^1 \psi^2(u)du$$

then $\sqrt{n}\, S_n^* = \sum_{t=1}^{n} Z_t^*$ with variance $\sigma_t^2 = \sum_{t=1}^{n} \text{Var}(Z_t^*)$.

For every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\sigma_s^2} \sum_{t=1}^{n} \int_{|z_t^*| > \varepsilon\sigma_s} (Z_t^*)^2 \, dG(z_t^*)$$

$$= \lim_{n\to\infty} [\int_0^1 \psi^2(u)du]^{-1} \int_{\{u: |[f(x_t, \theta_0) - f(x_t, \theta_{(n)})]\psi(u)| > \varepsilon\sigma_s\}} \psi^2(u)du$$

$$\leq \lim_{n\to\infty} [\int_0^1 \psi^2(u)du]^{-1} \int_{\{u: |\psi(u)| > \varepsilon\xi_n\}} \psi^2(u)du \cdots\cdots (3.2)$$

where

$$\xi_n = \frac{\left[\sum_{t=1}^{n} [f(x_t, \theta_0) - f(x_t, \theta_{(n)})]^2\right]^{\frac{1}{2}}}{\max_{1\leq t\leq n} |f(x_t, \theta_0) - f(x_t, \theta_{(n)})|} [\int_0^1 \psi^2(u) \, du]^{\frac{1}{2}}.$$

Assumption C implies $\xi_n \longrightarrow \infty$ as $n \longrightarrow \infty$, and the upper bound (3.2) goes to zero. By the Lindeberg central limit theorem, the proof is completed.

In order to show the asymptotic normality of W_n, we obtain that because of

$$E_0\left[\left\{\frac{s(Y_t - f(x_t, \theta_{(n)}))}{s(Y_t - f(x_t, \theta_0))}\right\}^2 - 1\right]$$

$$= \int_{-\infty}^{\infty} \left\{\frac{s(y_t - f(x_t, \theta_{(n)}))}{s(y_t - f(x_t, \theta_0))}\right\}^2 g(y_t - f(x_t, \theta_0)) \, dy - 1$$

$$= \int_{-\infty}^{\infty} g(y_t - f(x_t, \theta_{(n)})) \, \mathrm{dy} - 1,$$

$$E_0\left[\left\{\frac{s(Y_t - f(x_t, \theta_{(n)})) - s(Y_t - f(x_t, \theta_0))}{s(Y_t - f(x_t, \theta_0))}\right\}^2 + 2\left\{\frac{s(Y_t - f(x_t, \theta_{(n)})) - s(Y_t - f(x_t, \theta_0))}{s(Y_t - f(x_t, \theta_0))}\right\}\right] = 0$$

Consequently,

$$2E_0\left[\frac{s(Y_t - f(x_t, \theta_{(n)}))}{s(Y_t - f(x_t, \theta_0))} - 1\right] = -E\left[\left\{\frac{s(Y_t - f(x_t, \theta_{(n)})) - s(Y_t - f(x_t, \theta_0))}{s(Y_t - f(x_t, \theta_0))}\right\}^2\right].$$

Since $s(y)$ is absolutely continuous, so that $\lim_{h\to 0} \{s(y+h) - s(y)\}/h = s'(y)$ a.e and, furthermore, for every $h > 0$,

$$\int_{-\infty}^{\infty} \left|\frac{s(y+h) - s(y)}{h}\right| dy \cdots\cdots (3.3)$$

$$= \int_{-\infty}^{\infty} \left|\frac{1}{h} \int_y^{y+h} s'(t)dt\right| dy$$

$$\leq \int_{-\infty}^{\infty} [\frac{1}{|h|} \int_y^{y+h} |s'(t)|dt] \, dy$$

$$= \int_{-\infty}^{\infty} |s'(y)| \, dy$$

Note that

$$E_0(W_n)=2\sum_{t=1}^{n}E\left[\frac{s(Y_t-f(x_t,\underline{\theta}_{(n)}))}{s(Y_t-f(x_t,\underline{\theta}_0))}-1\right]$$

$$=-\sum_{t=1}^{n}E\left[\left\{\frac{s(Y_t-f(x_t,\underline{\theta}_{(n)})-s(Y_t-f(x_t,\underline{\theta}_0))}{s(Y_t-f(x_t,\underline{\theta}_0))}\right\}^2\right]$$

$$=-\sum_{t=1}^{n}[f(x_t,\underline{\theta}_0)-f(x_t,\underline{\theta}_{(n)})]^2\int_{-\infty}^{\infty}\left\{\frac{s(y_t-f(x_t,\underline{\theta}_{(n)}))-s(y_t-f(x_t,\underline{\theta}_0))}{f(x_t,\underline{\theta}_0)-f(x_t,\underline{\theta}_{(n)}))}\right\}^2 dy.$$

In particular, by (3.3), Under H_0,

$$\int_{\infty}^{\infty}\left\{\frac{s(y_t-f(x_t,\underline{\theta}_{(n)}))-s(y_t-f(x_t,\underline{\theta}_0))}{f(x_t,\underline{\theta}_0)-f(x_t,\underline{\theta}_{(n)})}\right\}^2 dy$$
$$\leq\int_{-\infty}^{\infty}\{s'(y_t-f(x_t,\underline{\theta}_0))\}^2\ dy.$$

therefore,

$$E_0(W_n)\leq-\frac{1}{4}\sum_{t=1}^{n}[f(x_t,\underline{\theta}_0)-f(x_t,\underline{\theta}_{(n)})]^2$$
$$x\int_{-\infty}^{\infty}\left[\frac{g'(y_t-f(x_t,\underline{\theta}_0))}{g(y_t-f(x_t,\underline{\theta}_0)}\right]^2 g(Y_t-f(x_t,\underline{\theta}_0))\ dy$$

then

$$E_0(W_n)\leq-\frac{1}{4}\ \mathrm{Var}(\sqrt{n}\ S_n^*)$$

where $\mathrm{Var}(\sqrt{n}\ S_n^*)=\sum_{t=1}^{n}[f(x_t,\underline{\theta}_0)-f(x_t,\underline{\theta}_{(n)})]^2$

$$x\int_{-\infty}^{\infty}\left[\frac{g'(y_t-f(x_t,\underline{\theta}_0))}{g(y_t-f(x_t,\underline{\theta}_0))}\right]^2 g(Y_t-f(x_t,\underline{\theta}_0))\ dy$$
$$=\sum_{t=1}^{n}[f(x_t,\underline{\theta}_0)-f(x_t,\underline{\theta}_{(n)})]^2\int_0^1\psi^2(u)du=\sigma_n^2$$

and that

$$\mathrm{Var}(W_n-\sqrt{n}\,S_n^*)=E_0(\{W_n-E(W_n)-\sqrt{n}\,S_n^*\}^2]\ \cdots\cdots(3.4)$$

$$\leq4\sum_{t=1}^{n}E_0\left[\left\{\frac{s(Y_t-f(x_t,\underline{\theta}_{(n)}))}{s(Y_t-f(x_t,\underline{\theta}_0))}\right\}-1-\frac{1}{2}[f(x_t,\underline{\theta}_0)-f(x_t,\underline{\theta}_{(n)})]\left\{\frac{g'(Y_t-f(x_t,\underline{\theta}_0))}{g(Y_t-f(x_t,\underline{\theta}_0)}\right\}\right]^2$$

$$\leq4\sum_{t=1}^{n}[f(x_t,\underline{\theta}_0)-f(x_t,\underline{\theta}_{(n)})]^2\left[\int_{-\infty}^{\infty}\left\{\frac{s(y_t-f(x_t,\underline{\theta}_{(n)}))-s(y_t-f(x_t,\underline{\theta}_0))}{f(x_t,\underline{\theta}_0)-f(x_t,\underline{\theta}_{(n)})}\right.\right.$$
$$\left.\left.-s'(y_t-f(x_t,\underline{\theta}_0))\right\}\right]^2 dy.$$

Then (3.3) implies that the right hand of (3.4) goes to zero as $n\longrightarrow\infty$. We have

W_n is asymptotically normal distribution (3.5)

with $$E_0(W_n) = -\frac{1}{4}\sigma^2$$

and $$\mathrm{Var}(W_n) = \sigma^2,$$

where $$\sigma^2 = \lim_{n=\infty} \sigma_n^2.$$

Lemma 3.2. Let Assumption A and B be satisfied.

Under the contiguous alternative hypothesis, S_j is asymptotically normally distributed

with $$E(S_j) = \lim_{n\to\infty} \mu_j$$

and

$$\mathrm{Var}(S_j) = \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \left[\frac{\partial f(x_t, \underset{\sim}{\theta}_0)}{\partial \underset{\sim}{\theta}_j}\right]^2 \int_0^1 \phi^2(u)\, du$$

$$j = 1, \cdots, p$$

where $\mu_j = \mathrm{Cov}(S_j, \sqrt{n}\, S_n^*)$

$$= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} [f(x_t, \underset{\sim}{\theta}_0) - f(x_t, \underset{\sim}{\theta}_{(n)})] \frac{\partial f(x_t, \underset{\sim}{\theta}_0)}{\partial \theta_j} \int_0^1 \phi\left(\frac{R_t}{n+1}\right)$$

$$x\left[-\frac{g'(y_t - f(x_t, \underset{\sim}{\theta}_0))}{g(y_t - f(x_t, \underset{\sim}{\theta}_0))}\right] dG(y)$$

Proof. By virtue of LeCam's third lemma (6), to establish the asymptotic normality of S_j under contiguous alternative hypotheses it suffices to show that

$$\lim_{n\to\infty} \max_{1\le t\le n} P\left[\left|\frac{g(y_t - f(x_t, \underset{\sim}{\theta}_{(n)}))}{g(y_t - f(x_t, \underset{\sim}{\theta}_0))} - 1\right| > \varepsilon\right] = 0$$

For every $\varepsilon > 0$,

$$\lim_{n\to\infty} \max_{1\le t\le n} P\left[\left|\frac{g(Y_t - f(x_t, \underset{\sim}{\theta}_{(n)}))}{g(Y_t - f(x_t, \underset{\sim}{\theta}_0))} - 1\right| > \varepsilon\right]$$

$$\le \lim_{n\to\infty} \max_{1\le t\le n} \frac{1}{\varepsilon} E\left[\left|\frac{g(Y_t - f(x_t, \underset{\sim}{\theta}_{(n)}))}{g(Y_t - f(x_t, \underset{\sim}{\theta}_0))} - 1\right|\right]$$

$$= \lim_{n\to\infty} \max_{1\le t\le n} \frac{1}{\varepsilon} |f(x_t, \underset{\sim}{\theta}_0) - f(x_t, \underset{\sim}{\theta}_{(n)})| \int_{-\infty}^{\infty} \left|\frac{g(y_t - f(x_t, \underset{\sim}{\theta}_{(n)})) - f(y_t, f(x_t, \underset{\sim}{\theta}_0))}{f(x_t, \underset{\sim}{\theta}_0) - f(x_t, \underset{\sim}{\theta}_{(n)})}\right| dy.$$

Now $$\left|\frac{g(y_t - f(x_t, \underset{\sim}{\theta}_{(n)})) - g(y_t - f(x_t, \underset{\sim}{\theta}_0))}{f(x_t, \underset{\sim}{\theta}_0) - f(x_t, \underset{\sim}{\theta}_{(n)})}\right|$$

$$\le \frac{1}{|f(x_t, \underset{\sim}{\theta}_0) - f(x_t, \underset{\sim}{\theta}_{(n)})|} \int_{y_t - f(x_t, \underset{\sim}{\theta}_0)}^{y_t - f(x_t, \underset{\sim}{\theta}_{(n)})} |g'(x)|\, dx.$$

We obtain that

$$\lim_{n\to\infty} \max_{1\le t\le n} \frac{1}{\varepsilon} |f(x_t,\underset{\sim}{\theta}_0)-f(x_t,\underset{\sim}{\theta}_{(n)})| \int_{-\infty}^{\infty} \left| \frac{g(y_t-f(x_t,\underset{\sim}{\theta}_{(n)}))-f(y_t-f(x_t,\underset{\sim}{\theta}_0))}{f(x_t,\underset{\sim}{\theta}_0)-f(x_t,\underset{\sim}{\theta}_{(n)})} \right| dy$$

$$\le \lim_{n\to\infty} \max_{1\le t\le n} \frac{1}{\varepsilon} |f(x_t,\underset{\sim}{\theta}_0)-f(x_t,\underset{\sim}{\theta}_{(n)})| \int_{-\infty}^{\infty} |g'(y)|dy \quad \cdots\cdots (3.5)$$

Since g has finite Fisher's information, the right hand side of (3.5) goes to zero as $n\longrightarrow\infty$. ///

3.2 Asymptotic Distribution of Test Statistic

We now consider the main result of this chapter, important theorem that establishes the asymptotic distribution of the test statistics under the null and alternative hypotheses. The preceding series of lemmas leads to the following one, which provides the key role in the construction of the asymptotic distribution of the test statistics Q. As mentioned before, the test statistics Q for testing the null hypothesis in (1.1) are defined as quadratic in signed rank order statistics. Writing S for $(S_1, S_2, \cdots, S_p)$, the test statistic Q is given by

$$Q=\underset{\sim}{S}'\ \Sigma_n^{-1}\ \underset{\sim}{S}$$

where Σ_n is the $p\times p$ covariance matrix of (S_i, S_j) with elements as

$$\mathrm{Cov}(S_i,S_j)=\frac{1}{n}\sum_{t=1}^{n} \frac{\partial f(x_t,\underset{\sim}{\theta}_0)}{\partial \underset{\sim}{\theta}_i}\ \frac{\partial f(x_t,\underset{\sim}{\theta}_0)}{\partial \underset{\sim}{\theta}_j} \int_0^1 \phi^2(u)\ du.$$

Observe that Q is well defined since both Σ_n^{-1} and its limit as $n\to\infty$ exist with Assumptions A, B and C. Furthermore, it easily seen that by Lemma 2.1, 3.1 and under H_0, $(S_j, \sqrt{n}\ S_n^*)$ tend to the bivariate normal distribution with correlation coefficients γ_j, $j=1,\cdots,p$, where

$$\gamma_j=\frac{\mathrm{Cov}(S_j,\ \sqrt{n}\ S_n^*)}{\{\mathrm{Var}\ S_j\}^{\frac{1}{2}}\ \{\mathrm{Var}(\sqrt{n}\ S_n^*)\}^{\frac{1}{2}}}$$

$$=\frac{\sum_{t=1}^{n}[(fx_t,\underset{\sim}{\theta}_0)-f(x_t,\underset{\sim}{\theta}_{(n)})]\ \frac{\partial f(x_t,\underset{\sim}{\theta}_0)}{\partial\underset{\sim}{\theta}_j}\int_0^1 \phi^2(u)\ \psi(u)\ du}{\left[\sum_{t=1}^{n}\left[\frac{\partial f(x_t,\underset{\sim}{\theta}_0)}{\partial\underset{\sim}{\theta}_j}\right]^2\int_0^1\phi^2(u)du\right]^{\frac{1}{2}}\left[\sum_{t=1}^{n}[f(x_t,\underset{\sim}{\theta}_0)-f(x_t,\underset{\sim}{\theta}_{(n)})]^2\int_0^1\psi^2(u)\ du\right]^{\frac{1}{2}}}$$

Combining the test statistics Q with Lemma 2.1 and Lemma 3.2, we have the following result.

Theorem 3.6. (a) Under Assumptions A, B and the null hypothesis, Q has asymptotically the central chi-square distribution with p degrees freedom.

(b) Under Assumptions A, B, C and the contiguous alternatives Q has asymptotically the noncentral chi-square distribution with p degrees of freedom and noncentrality parmater

$$\lambda=\frac{1}{2}\ \underset{\sim}{\mu}'\ \Sigma^{-1}\ \underset{\sim}{\mu}$$

where $\underset{\sim}{\mu}$ is the $p\times 1$ vector of μ_j (Lemma 3.2) and the limit of a covariance matrix is the matrix Σ formed by the limit of each entry in the matrix.

Proof. The proof follows immediately from Lemmas 2.1, 3.2 and LeCam's second Lemma (6). ///

Let $X_p^2(\alpha)$ be the upper $100\alpha\%$ point of chi-square distribution function with p degrees of freedom. From the discussion above, we have the following test procedure for large n:

Reject or Accept H_0: $\underset{\sim}{\theta}=\underset{\sim}{\theta}_0$ according as

$$Q\geq \text{or} < X_p^2(\alpha)$$

where $\alpha(Q<\alpha<1)$ is the desired level of significance of the test.

4. Asymptotic Efficiency of Test Statistics

4.1. Asymptotic efficiency of Q test.

In this section we consider the comparison of two test procedures due to Pitman (11). As mentioned before, if two tests of the same size of the same statistical hypothesis have asymptotically noncentral chi-square distributions, the asymptotic relative efficiency of the second test with respect to the first is given by the ratio of their noncentrality parmeters.

For the first test statistic $Q=\underset{\sim}{S}'\ \Sigma_n^{-1}\ \underset{\sim}{S}$, it has been shown that under a sequence of contiguous alternatives H_1: $\underset{\sim}{\theta}=\underset{\sim}{\theta}_{(n)}$ tending to the null hypothesis at a certain rate, Q has an asymptotic noncentral chi-square distribution with the noncentrality parameter

$$\lambda_Q = \frac{1}{2} \underset{\sim}{\mu}' \Sigma^{-1} \underset{\sim}{\mu} \cdots\cdots\cdots\cdots (4.1)$$

where Σ is the $p \times p$ matrix with the (i, j)th elements as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \frac{\partial f(x_t, \underset{\sim}{\theta}_0)}{\partial \underset{\sim}{\theta}_i} \frac{\partial f(x_t, \underset{\sim}{\theta}_0)}{\partial \underset{\sim}{\theta}_j} \int_0^1 \phi^2(u)\, du,$$

and $\underset{\sim}{\mu}$ is the $p \times 1$ vector with the j-th element μ_j as

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} [f(x_t, \underset{\sim}{\theta}_0) - f(x_t, \underset{\sim}{\theta}_{(n)})] \frac{\partial f(x_t, \underset{\sim}{\theta}_0)}{\partial \underset{\sim}{\theta}_j} \int_0^1 \phi(u)\, \psi(u)\, du.$$

On the other hand, it is well known (see (8)) that under certain regularity conditions, the classical least squares estimator, $\underset{\sim}{\theta}_n^*$, has an asymptotic normal distribution in the sense that

$$\sqrt{n}\, (\underset{\sim}{\theta}_n^* - \underset{\sim}{\theta}) \xrightarrow{D} N(\underset{\sim}{0}, \sigma^2 \Sigma^*)$$

where $\Sigma^{*^{-1}}$ is a $p \times p$ matrix with the (i, j)th element as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \frac{\partial f(x_t, \underset{\sim}{\theta})}{\partial \underset{\sim}{\theta}_i} \frac{\partial f(x_t, \underset{\sim}{\theta})}{\partial \underset{\sim}{\theta}_j} \cdots\cdots\cdots\cdots (4.2)$$

and $\underset{\sim}{\theta}$ is the true parameter of $\underset{\sim}{\theta}_n^*$. The second test statistic Q^* based on the quadratic function in then least squares estimators $\underset{\sim}{\theta}_n^*$ is therefore given by

$$Q^* = \frac{n}{\sigma^2} \underset{\sim}{\theta}_n^{*\prime} \Sigma^{*^{-1}} \underset{\sim}{\theta}_n^*$$

Under the given alternative hypothesis of the form $\underset{\sim}{\theta} = \underset{\sim}{\theta}_{(n)}$, Q^* has a symptotic noncentral chi-square distribution with p degrees of freedom and noncentrality parameter

$$\lambda_Q^* = \frac{1}{2\sigma^2} \underset{\sim}{\mu}^{*\prime} \Sigma^{*^{-1}} \underset{\sim}{\mu}^* \cdots\cdots\cdots\cdots (4.3)$$

where $\underset{\sim}{\mu}^*$ is the $p \times 1$ vector $\lim_{n\to\infty} \sqrt{n}\, \underset{\sim}{\theta}_{(n)}$.

From (4.1) and (4.3) the asymptotic efficiency of the Q test relative to Q^* test is therefore

$$e_{Q/Q^*} = \sigma^2 \{\underset{\sim}{\mu}' \Sigma^{-1} \underset{\sim}{\mu}\} / \{\underset{\sim}{\mu}^{*\prime} \Sigma^{*^{-1}} \underset{\sim}{\mu}^*\}$$

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