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Maximum Principles for Functionals Associated with Semilinear Elliptic and Parabolic Problems

by

Jeong-Seon Baek

Seoul National University, Seoul (151) Korea

1. Introduction

In [1] Payne and Stakgold proved the following result: Let u be a solution of

(1.1) $$\triangle u+f(u)=0 \text{ in } D\subseteq R^n,\ u=0 \text{ on } \partial D.$$

If the boundary ∂D has nonnegative mean curvature, then the functional $|\text{grad } u|^2+2\int_0^u f(s)ds$ attains its maximum at a point where grad $u=0$.

In this paper we give an extension of this theorem to the case that u is a solution of

(1.2) $$\triangle u+f(u,\ \frac{1}{2}|\text{grad } u|^2)=0 \quad \text{in } D.$$

In this case the corresponding functional is $F(u,\ \frac{1}{2}|\text{grad } u|^2)$ with $F(u,v)$ a function satisfying that

$$F_u=fF_v \text{ and } F_v>0$$

for $u\in R$ and $v>0$. Our result is then the following:

Let u be a solution of (1.2). Then the above functional attains its maximum on ∂D or at a point where grad $u=0$. If, moreover, the boundary value of u is constant, then $F(u,\ \frac{1}{2}|\text{grad } u|^2)$ attains its maximum at a point x such that either

(a) grad $u(x)=0$; or

(b) grad $u(x)\neq 0$ and the mean curvature of ∂D is strictly negative.

Another extension to the parabolic problems is given:

If $u(x,t)$ is a solution of

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$$(1.3)\qquad \triangle u-u_t+f(u,\ \tfrac{1}{2}|\mathrm{grad}\ u|^2)=0 \text{ in } D\times(O,T).$$

Then the functional $F(u,\ \frac{1}{2}|\mathrm{grad}\ u|^2)$ attains its maximum at (x,t) such that (i) $x\in\partial D$, (ii) $t=0$ or (iii) grad $u=0$ at (x,t).

If, moreover, u is constant on $\partial D\times(O,T)$, then, instead of (i), we may have (iv) $x\in\partial D$ and the mean curvature of ∂D at x is strictly negative.

2. Preliminary iemmas

Throughout this paper we assume appropriate conditions on the smoothness of functions and the boundary of a bounded domain in the n dimensional Euclidean space R^n. We make use of the notations F_u, F_v and w_i to indicate the partial derivatives of $F(u,v)$ and $w(x)$ (or $w(x,t)$) with respect to the corresponding variables u,v and x_i. As usual, we adopt the summation convention on repeated indices. For $w(x)$ or $w(x,t)$, we denote by $\triangle w$ and grad w the Laplacian and the gradient of w with respect to the space variable x.

Let $F(u,v)$ be a continuous function on $R\times \bar{R}_+$ which is of class C^2 in $R\times R_+$. Our result starts with the following two technical lemmas.

Lemma 1. Suppose that $u(x)$ is a solution of (1.2). Then the function $w(x)=F(u(x),\ \frac{1}{2}|\mathrm{grad} u(x)|^2)$ satisfies the inequality

$$(2.1)\qquad \begin{aligned}&\triangle w-\mathrm{L}\cdot\mathrm{grad}\ w/F_v^2|\ \mathrm{grad}\ u|^2\geq -F_u f+F_u^2/F_v+|\mathrm{grad}\ u|^2\times\\ &(F_u f_v-F_v f_u+F_{uu}-2F_{uv}(F_u/F_v)+F_{vv}(F_u/F_v)^2)\end{aligned}$$

at points where grad $u\neq 0$ and $F_v>0$, with some bounded vector L.

Proof. Setting $v(x)=\frac{1}{2}|\mathrm{grad}\ u(x)|^2$, we get, by successive differentiation, the identities

$$(2.2)\qquad \begin{aligned}&w_k=F_u u_k+F_v v_k,\\ &\triangle w=F_u\triangle u+F_v\triangle v+F_{uu}u_k u_k+2F_{uv}u_k v_k+F_{vv}v_k v_k,\\ &v_k=u_i u_{ik},\\ &\triangle v=u_{ik}u_{ik}+u_i\triangle u_i.\end{aligned}$$

Taking account of the Schwarz's inequality, (2.3) $$u_{ik}u_{ik}u_ju_j \geq u_iu_{ik}u_ju_{jk},$$

from (1.2), we deduce that

(2.4) $$\triangle v \geq v_kv_k/u_ju_j - f_uu_ku_k - f_vu_kv_k.$$

Substituting $v_k=(w_k-F_uu_k)/F_v$ in (2.2) and (2.4), we deduce (2.1), with $F_vw_k-2F_uF_vu_k+|\text{grad } u|^2(-F_vf_vu_k+2F_{uv}u_k+F_{vv}w_k-2F_uF_vu_k)$ as the k-th component of L. This completes the proof. ///

In the case of parabolic problems we have also an analogue of the preceding lemma.

Lemma 2. Suppose that $u(x,t)$ is a solution of (1.3). Then the function $w(x,t)=F(u, \frac{1}{2}|\text{grad } u|^2)$ satisfies the inequality (2.1) in which the term $\triangle w$ is replaced by $\triangle w-w_t$.

Proof. Straightforward calculation leads us to

$$w_t=F_uu_t+F_vv_t,$$
$$\triangle w-w_t=F_u(\triangle u-u_t)+F_v(\triangle v-v_t)+F_{uu}u_ku_k+2F_{uv}u_kv_k+F_{vv}v_kv_k,$$
$$\triangle v-v_t=u_{ik}u_{ik}+u_i(\triangle u-u_t)_i.$$

Now, following the proof of lemma 1, we come by (2.1) with $\triangle w$ replaced by $\triangle w-w_t$. ///

3. Maximum Principles

In this section we assume that

(3.1) $$F_u=fF_v,$$

(3.2) $$F_v>0 \text{ for } u\in R,\ v>0.$$

Taking into account that $f_u=(F_{uu}F_v-F_uF_{uv})/F_v^2$ and that $f_v=(F_{uv}F_v-F_uF_{vv})/F_v^2$, with a simple calculation, we can see that the expression on the right hand side of (2.1) vanishes to zero, for $u\in R$ and $v>0$.

Now let $u(x)$ be a solution of (1.2), we set

$$w(x)=F(u(x), \frac{1}{2}|\text{grad } u(x)|^2).$$

Then by Lemma 1 we have

(3.3) $$\triangle w - L\cdot \text{grad}\ w/F_v^2|\text{grad}\ u|^2 \geq 0,$$

whenever grad $u \neq 0$.

Let U be the set of all x in D such that w attains its maximum at x and grad $u(x) \neq 0$. Then, by the ordinary maximum principle, it follows that U is an open set. If U is not empty, then the boundary of U must contain a point on the boundary of D or a point where grad $u=0$, hence it follows that w attains its maximum on the boundary of D or at a point where grad $u=0$, which holds also in the case when U is empty. We sum up in

Theorem 1. Let u be a solution of (1.2). Suppose that $F(u,v)$ satisfies (3.1) and (3.2). Then $F(u, \frac{1}{2}|\text{grad}\ u|^2)$ attains its maximum on the boundary of D or at a point where grad $u=0$.

Remark. Suppose that U is not empty and that grad $u=0$ on $\bar{U}$. Then the boundary of U is contained in that of D, and hence $U=D$, since D is a domain. Thus w is constant and grad $u \neq 0$ in $\bar{D}$. It is very interesting to distinguish the problems for which this situation may happen. But this is not the purpose of the present paper.

Now we continue the arguement preceding Theorem 1.

Suppose that w attains its maximum at a point x on the boundary of D with grad $u(x) \neq 0$. Applying the second maximum principle of Hopf, we deduce that

(a) w is constant in a neighborhood of x; or

(b) $\partial w/\partial n > 0$ at x,

where n denotes the outward unit normal to the boundary of D.

In case (a), U is not empty, hence, by the preceding remark, we have two possibilities:

(a1) w is constant and grad $u \neq 0$ in $\bar{D}$; or

(a2) w attains its maximum at a point where grad $U=0$.

If the boundary value of u is constant, then the possibility (a1) dose not arise. If, moreover, the boundary of D has nonnegative mean curvature, then (b) cannot hold. In fact, we have

$$\begin{aligned}\partial w/\partial n = w_k n_k &= (F_u u_k + F_v v_k) n_k = F_v (f u_k + u_i u_{ik}) n_k \\ &= F_v (u_i u_{ik} - u_k \triangle u) n_k = -F_v |\text{grad}\ u|^2 H,\end{aligned}$$

by noting that the mean curvature H is equal to

$$\pm(u_i u_{ik} u_k - u_{ii} u_k u_k)/|\text{grad } u|^3$$

according as $n=\pm\text{grad } u/|\text{grad } u|$. Thus (b) implies that $H(x)<0$.

Summing up, we have

Theorem 2. In addition to the hypotheses of Theorem 1, suppose that u is constant on the boundary of D. Then $w=F(u, \frac{1}{2}|\text{grad } u|^2)$ attains its maximum at a point where grad $u=0$, or at a point on the boundary where the mean curvature is strictly negative. In particular, if the mean curvature of the boundary of D is nonnegative, then w has its maximum at a point where grad $u=0$.

Remark. In U, the level surfaces of u have the zero mean curvature, as can be shown by the arguement preceding Theorem 2.

With the help of Lemma 2 and the ordinary maximum principle for parabolic inequalities, we can prove the following maximum principle for the problem (1.3).

Theorem 3. Let $u(x,t)$ be a solution of (1.3). Suppose that $F(u,v)$ satisfies (3.1) and (3.2). Then $w(x,t)=F(n, \frac{1}{2}|\text{grad } u|^2)$ attains its maximum at a point where grad $u=0$, or at the parabolic boundary of $D\times(O,T)$. If u is constant on the boundary of D at every instant of time and if the boundary of D has nonnegative mean curvature, then w attains its maximum at a point where grad $u=0$ or $t=0$.

Remark 1. Without any essential change of arguements, we can investigate the problem, with A assumed to be a constant vector,

$$\triangle u + A\cdot\text{grad } u - u_t + f(u, \frac{1}{2}|\text{grad } u|^2)=0 \text{ in } D\times(O,T),$$

and obtain the same result as in the preceding theorem.

For the equations

$$a^{ij}u_{ij}+a^i u_i+f(u, \frac{1}{2}a^{ij}u_i u_j)=0 \text{ in } D, \text{ or}$$

$$a^{ij}u_{ij}+a^i u_i-u_t+f(u, \frac{1}{2}a^{ij}u_i u_j)=0 \text{ in } D\times(O,T),$$

we can obtain, with the corresponding functional $F(u, \frac{1}{2}a^{ij}u_i u_j)$, similar results to the preceding theorems. Here, a^{ij} is a positive definite constant matrix and a^i is a constant vector. In the case when the coefficients depend on x in D, then similar result may be obtained with some technique of Riemannian geometry, under some additional assumptions. It is a very difficult problem to distinguish out much more general classes of equations of elliptic or parabolic type for which analogues of the preceding results could be obtained; for instance, in case of the equations

$$\triangle u + f(x, u) = 0 \text{ in } D \subset R^n$$

we have as yet no knowledge of analogues of Theorem 1 and 2.

References

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