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On the Space S_n and the Fourier Transforms of S'_n

by

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1. Introduction

It is a well known fact that the creation of the theory of distributions by the French mathematician Laurent Schwartz ([12]) is an event of great significance in the history of mordern mathematics. In particular, this theory has an important effect on the Fourier transform. Also, it is a well known fact that the applications of the Fourier transforms to the theory of differential equations have rendered helps to development of many parts of science, so that vigorous research progress about the distribution and the Fourier transforms in the functional analysis has been accomplished.

The purpose of this paper is to prove some properties of the rapidly decreasing function space S_n (cf. section 3) and the Fourier transforms in S'_n (cf. section 4), which is a dual space of S_n.

In detail, the contents of this paper is as follows. In §2, we explain the terminologies and prove the basic properties used in later sections. In particular, we prove that "$\mathscr{D}(R^n)$ is dense in $C^\infty(R^n)$ (Proposition 2.2)." In §3, we prove that "the topology of S_n can not be induced by any translation invariant metric topology ([15]), which turns the Fourier transfrom (see Property 2) into an isometry of S_n onto S_n (Theorem 3.4)." In §4, we prove the Lemma 4.4 and Theorem 4.5. In particular, we assert the followings in Theorem 4.5: "Let u be a tempered distribution with compact support K such that $\hat{u}$ (Fourier transforms of u) is bounded. Then $\Psi u=0$ for every $\Psi\in C^\infty(R^n)$, which vanishes on K if $n=1,2$.

Throughout this paper, the letters C and R stand for the field of complex numbers and for the field of real numbers, respectively.

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2. Preliminaries

In this paper, the term *multi-index* means an ordered n-tuple

$$\alpha=(\alpha_1, \cdots, \alpha_n)$$

of nonnegative integers $\alpha_i (i=1, 2, \cdots, n)$.
If $\alpha=(\alpha_1, \cdots, \alpha_n)$ and $\beta=(\beta_1, \cdots, \beta_n)$ are multi-indices, then

$$|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n,$$
$$D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n} \quad (D_j=\frac{\partial}{\partial x_j}),$$
$$\beta\leq\alpha \text{ means } \beta_i\leq\alpha_i \text{ for } 1\leq i\leq n,$$
$$\alpha\pm\beta=(\alpha_1\pm\beta_1, \cdots, \alpha_n\pm\beta_n).$$

If $x, y\in R^n$, then

$$|x|=\{x_1^2+\cdots+x_n^2\}^{\frac{1}{2}},$$
$$x.y=x_1y_1+\cdots+x_ny_n,$$

where $x=(x_1, \cdots, x_n)$ and $y=(y_1, \cdots, y_n)$.
Moreover the monomial x^α is defined by

$$x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}.$$

We shall also use the following notations in this paper.

$$C^\infty(R^n)=\{f: R^n\longrightarrow C \mid \text{for every multi-index } \alpha, D^\alpha f \text{ is continuous}\}.$$
$$\mathcal{D}_K(R^n)=\{f\in C^\infty(R^n) \mid \operatorname{supp}(f)\subset K\},$$

where K is a compact subset of R^n and $\operatorname{supp}(f)$ is the support of f.

$$\mathcal{D}(R^n)=\bigcup_{K:\text{compact in } R^n} \mathcal{D}_K(R^n).$$

$L^p(R^n)=\{f;\text{measurable} \mid \{\int_{R^n}|f(x)|^p dm_n(x)\}^{\frac{1}{p}}<\infty\}$, where m_n is the *normalized Lebesgue measure* on R^n which is defined by

$$dm_n(x)=(2\pi)^{-\frac{n}{2}}dx_1\cdots dx_n=(2\pi)^{-\frac{n}{2}}dx.$$

Proposition 2.1. *If $1\leq p<\infty$, then $\mathcal{D}(R^n)$ is dense in $L^p(R^n)$.*

Proof. We shall use the following results in our proof.

1°. (*The dominated convergence theorem* ([11])) Let $\{f_n\}$ be a sequence of complex measurable functions on R^n such that

$$\lim_{m\to\infty} f_m(x)=f(x).$$

exists for every $x\in R^n$.

If there is a function $g\in L^p(R^n)$ $(1\leq p<\infty)$ such that

$$|f_m(x)|^p\leq|g(x)|^p \quad (m=1,2,3,\cdots;x\in R^n),$$

then $f\in L^p(R^n)$,

$$\lim_{m\to\infty} \int_{R^n}|f_m(x)-f(x)|^p\, dm_n(x)=0$$

and

$$\lim_{m\to\infty} \int_{R^n}|f_m(x)|^p\, dm_n(x)=\int_{R^n}|f(x)|^p\, dm_n(x).$$

2°. (*Lusin's Theorem* ([11])) Let f be measurable function such that there exists a subset $A\subset R^n$ with

$$x\notin A \Longrightarrow f(x)=0,\quad m_n(A)<\infty,$$

then there exists a $g\in\mathscr{D}(R^n)$ such that

$$m_n(\{x\in R^n|f(x)\neq g(x)\})<\varepsilon,$$

where ε is a given positive number.
Furthermore, we may arrange it so that

$$\sup_{x\in R^n}|g(x)|\leq\sup_{x\in R^n}|f(x)|.$$

3°. ([11]) Let f be measurable. There exist simple measurable functions s_m such that

(a) $0\leq s_1\leq s_2\leq\cdots\leq f$.
(b) $s_m(x)\longrightarrow|f(x)|$ as $m\to\infty$, for every $x\in R^n$.

In order to prove our proposition we have two steps;

Step I. $S=\{s|s$ is measurable and $m_n(\{x\in R^n)|s(x)=0\})<\infty\}$. We want to prove that S is dense in $L^p(R^n)$.

Since it is clear that $S\subset L^p(R^n)$, let us suppose $f\geqq 0$ and $f\in L^p(R^n)$.

Then by 3° above, there exists a sequence of simple measurable functions s_m such that

$$0\leq s_1\leq s_2\leq s_3\leq\cdots\leq f,$$
$$\lim_{m\to\infty} s_m(x)=f(x),\ \text{for all } x\in R^n.$$

Hence, for all $x \in R^n$

$$|f(x)-s_m(x)|^p \leq (f(x))^p.$$

Thus, by 1° above, we have

$$\lim_{m\to\infty} (\int_{R^n} |f(x)-s_m(x)|^p dm_n(x))^{\frac{1}{p}}=0.$$

This means that in the topology of $L^p(R^n)$ f is in the closure of S.

Step Ⅱ. In order to prove our assertion, it suffices to verify that in the topology of $L^p(R^n)$ $\mathscr{D}(R^n)$ is dense in S, as S is dense in $L^p(R^n)$.

By 2° above, for each $s \in S$ and a positive number ε, there exists $g \in \mathscr{D}(R^n)$ such that

$$m_n(\{x \in R^n | s(x) \neq g(x)\}) < \varepsilon.$$

Moreover, in this situation

$$\sup_{x\in R^n} |g(x)| \leq \sup_{x\in R^n} |s(x)| = \|s\|_\infty.$$

Thus,

$$|g(x)-s(x)|^p \leq 2^p \|s\|_\infty, \text{ for all } x \in R^n, \cdots\cdots\cdots\cdots (A)$$

and

$$\{\int_{R^n} |g(x)-s(x)|^p dm_n(x)\}^{\frac{1}{p}} \leq 2\varepsilon^{\frac{1}{p}} \|s\|_\infty.$$

Thus $\mathscr{D}(R^n)$ is dense in S, in the L^p-topology.

In fact, for each $f \in L^p(R^n)$ there exists $s \in S$ such that $\|f-s\|_q < \frac{\varepsilon}{2}$.

By Setp Ⅱ or (A) there exists $g_\varepsilon \in \mathscr{D}(R^n)$ with $\|s-g_\varepsilon\| < \frac{\varepsilon}{2}$.

Thus

$$\|f-g_\varepsilon\|_p \leq \|f-s\|_p + \|s-g_\varepsilon\|_p < \varepsilon,$$

which implies that f is in the closure of $\mathscr{D}(R^n)$ in the L^p-topology. ///

The topology of $C^\infty(R^n)$ is defined as follows. Take compact subsets K_i of R^n such that

(a) $R^n = \bigcup_{i=1}^{\infty} K_i$.

(b) for all $i \geq 1$, K_i is contained in the interior of K_{i+1}.

Put

$$\rho_N(f) = \max\{|\mathscr{D}^\alpha f(x)| \,|\, x \in K_N, \ |\alpha| \leq N\}$$

$$(\rho_N \text{ is a seminorm of } C^\infty(R^n)) \text{ for each } f \in C^\infty(R^n)$$

and

$$V_N = \{f \in C^\infty(R^n) \mid \rho_N(f) < \frac{1}{N},\ N = 1, 2, \cdots\}.$$

We take $\{V_N \mid N = 1, 2, \cdots\}$ as a local base of $C^\infty(R^n)$; that is, every neighborhood of zero contains a member of $\{V_N\}$. Then $C^\infty(R^n)$ becomes a Fréchet space with the Heine-Borel property ([10]).

In this topology, the metric between f and g $(f, g \in C^\infty(R^n))$ is defined by

$$d(f, g) = \sum_{N=1}^{\infty} \frac{2^{-N} \rho_N(f-g)}{1 + \rho_N(f-g)}$$

We note that to prove that $d(f, g)$ is the metric on $C^\infty(R^n)$, we have to use following;

$$b > a > 0 \Longrightarrow \frac{a}{1+a} < \frac{b}{1+b}$$

Proposition 2.2. *$\mathscr{D}(R^n)$ is dense in the Fréchet space $C^\infty(R^n)$.*

Proof. Take $f \in C^\infty(R^n)$ and a positive integer N. Then there exists $\Psi_N \in \mathscr{D}(R^n)$ such that $\Psi_N \mid K_N = 1$, where K_N is a compact subset of R^n defined as above. In fact, let $\mathscr{U}$ be an open covering in R^n. Then there is a locally finite partition of unity $\{\varphi_i\}$ $(\varphi_i \in \mathscr{D}(R^n))$ in R^n subordinate to the $\mathscr{U}$. In this case, to each compact $K_N \subset R^n$ there correspond an integer m and an open subset W(of R^n)$\subset K_N$ such that

$$\varphi_1(x) + \cdots + \varphi_m(x) = 1 \text{ for all } x \in W.$$

Therefore, if we put

$$\Psi_N(x) = \varphi_1(x) + \cdots + \varphi_m(x),$$

then $\Psi_N|_{K_N} = 1$ and supp $(\Psi_N) =$ supp $(\varphi_1) \cup \cdots \cup$ supp (φ_m) ([10]).
We put

$$g_N = f \cdot \Psi_N \in \mathscr{D}(R^n),$$

then

$$f = g_N \text{ on } K_N.$$

Hence

$$f - g \in V_N,$$

and since $V_N \to 0$ as $N \to \infty$, in the topology of $C^\infty(R^n)$. i.e., $g_N \to f$. Thus $\mathscr{D}(R^n)$ is dense in the Fréchet space $C^\infty(R^n)$. ///

For each $f \in \mathscr{D}_K(R^n)$, let us introduce the seminorm of f,

$$\|f\|_N = \max\{|(D^\alpha f)(x)| \mid x \in K, \ |\alpha| \leq N\}.$$

By this seminorms $\mathscr{D}_K(R^n)$ be a Fréchet space ([10]). We shall denote this Fréchet space topology of $\mathscr{D}_K(R^n)$ by τ_K.

Definition 2.3. (a) β is the collection of all convex balanced sets $W \subset \mathscr{D}(R^n)$ (for all $\alpha \in R$ with $|\alpha| \leq 1$, $\alpha W \subset W$) such that $W \cap \mathscr{D}_K(R^n) \in \tau_K$ for every compact $K \subset R^n$.

(b) τ is the collection of all unions of the from $\phi + W$, with $\phi \in \mathscr{D}(R^n)$ and $W \in \beta$.

Then τ is a topology in $\mathscr{D}(R^n)$, and β is the local base for τ ([10]). Hearafter, by $\mathscr{D}(R^n)$ we mean a topological vector space with topology τ. In the topological vector space $\mathscr{D}(R^n)$, the following hold ([10]).

1°. $\mathscr{D}R^n$ is locall convex.

2°. The topology τ_K of $\mathscr{D}(R^n)$ coincides with the subspace topology that $\mathscr{D}_K$ inherits form $\mathscr{D}(R^n)$, where K is an any compact subset of R^n.

3°. $\mathscr{D}(R^n)$ has the Heine-Borel property.

4°. Every Cauchy sequence converges in $\mathscr{D}(R^n)$.

Definition 2.4. A linear fuctional on $\mathscr{D}(R^n)$ which is continuous with respect to the topology τ described in Definition 2.3 is called a *distribution* in R^n. The space of all distribution in R^n is denoted by $\mathscr{D}'(R^n)$.

The topology of $\mathscr{D}'(R^n)$ is the *weak*-topology* induced by $\mathscr{D}(R^n)$.

Example 2.5. We shall illustrate some distributions ([3],[12],[13],[14],[15],).

(a) For each multi-index α and each $\Lambda \in \mathscr{D}'(R^n)$, $D^\alpha \Lambda$ is also an element of $\mathscr{D}'(R^n)$. That is, for each $\phi \in \mathscr{D}(R^n)$

$$D^\alpha \Lambda(\phi) = (-1)^\alpha \Lambda(D^\alpha \phi).$$

Note that $D^\alpha \phi \in \mathscr{D}(R^n)$ for $\phi \in \mathscr{D}(R^n)$. If β is a multi-index, then

$$D^\alpha(D^\beta \Lambda) = D^{\alpha+\beta}\Lambda = D^\beta(D^\alpha \Lambda) \in \mathscr{D}'(R^n).$$

(b) For each $f \in C^\infty(R^n)$, Λ_f is defined by

$$\Lambda_f(\phi) = \int_{R^n} f(x) \cdot \phi(x) \, dm_n(x) \quad [\phi \in \mathscr{D}(R^n)].$$

Then $\Lambda_f \in \mathscr{D}'(R^n)$.

(c) For each $f \in C^\infty(R^n)$ and $\Lambda \in \mathscr{D}'(R^n)$, $f\Lambda$ is defined by $(f\Lambda)(\phi) = \Lambda(f\phi)$, for each $\phi \in \mathscr{D}(R^n)$.

Then $f\Lambda$ is a distribution in R^n.

We have to note that if $\Lambda \in \mathscr{D}'(R^n)$, then there are a nonnegative integer N and a constant $C<\infty$ such that the inequality

$$|\Lambda\phi| \leq C\|\phi\|_N$$

holds for every $\phi \in \mathscr{D}_K(R^n)$, where K is a compact subset of R^n. If Λ is such that one N will do for all compact subsets K of R^n(but not necessarily with the same C), then the smallest such N is called the *order* of Λ. If no N will do for all K, then Λ is said to have *infinite order*.

Definition 2.6. Suppose $\Lambda \in \mathscr{D}'(\mathscr{D}^n)$. If W is an open subset of R^n and if $\Lambda\phi=0$ for every $\phi \in \mathscr{D}(W)$, we say that Λ vanishes in W. Let W be the union of all open $W \subset R^n$ in which Λ vanishes. The complement of W (relative to R^n) is the *support* of Λ denoted by S_Λ.

We have the following properties about $\Lambda \in \mathscr{D}'(R^n)$ ([10]).

1°. If S_Λ is empty, then $\Lambda=0$.

2°. If S_Λ is a compact subset of R^n, then Λ has finite order. Furthermore, Λ extends in a unique way to a continuous linear functional on $C^\infty(R^n)$. In fact, for each $f \in C^\infty(R^n)$ there exists a sequence $\{f_i \mid i=1,2,\cdots,\ f_i \in \mathscr{D}(R^n)\}$ such that in the Fréchet space $C^\infty(R^n)$ $f_i \longrightarrow f$ by Proposition 2.2, and

$$\Lambda(f) = \lim_{i\to\infty} \Lambda(f_i)$$

in R (or C).

3. The Space S_n

The *convolution* of two functions f and g on R^n is defined by

$$\begin{aligned}(f*g)(x) &= \textstyle\int_{R^n} f(y-x)g(y)\ dm_n(y)\\ &= \textstyle\int_{R^n} f(y)g(x-y)\ dm_n(y)\end{aligned}$$

whenever the integral exist.

For each $t \in R^n$, the *character* e_t is the function defined by

$$e_t(x) = e^{it\cdot x} = \exp\ \{i(t_1x_1 + \cdots\cdots t_nx_n)\}$$

for all $x=(x_1,\cdots,x_n) \in R^n$, where $t=(t_1,\cdots,t_n)$.

It follows that for each $t \in R^n$, $e_t \in C^\infty(R^n)$.

Definition 3.1. (a) The *Fourier transform* of a function $f\in L^p(R^n)$ is the function $\hat{f}$ defined by

$$\hat{f}(t)=\int_{R^n}f(x)e_{-t}(x)\ dm_n(x)$$
$$=\int_{R^n}f(x)e^{-itx}\ dm_n(x)$$

([2], [4], [10]).

Also the term "Fourier transfrom" is often used for the mapping that takes f to $\hat{f}$.

(b) For each multi-index $\alpha=(\alpha_1,\cdots,\alpha_n)$

$$D_\alpha=\left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

By our definition, it is clear that

$1°$. $\hat{f}(t)=(f*e_t)(0)$
$$=\int_{R^n}f(y)e_t(-y)\ dm_n(y)$$
$$=\int_{R^n}f(y)e^{-it\cdot y}\ dm_n(y).$$

$2°$. $D_\alpha e_t=t^\alpha e_t$ $(t^\alpha=t_1^{\alpha_1}\cdots t_n^{\alpha_n})$.

Definition 3.2. Suppose $f\in C^\infty(R^n)$. f is said to be a *rapidly decreasing function* if

$$\sup_{|\alpha|\le N}\ \sup_{x\in R^n}\{(1+|x|^2)^N|(D_\alpha f)(x)|\}=\rho_N(f)<\infty$$

for $N=0,1,2,\cdots\cdots$.

The set of all rapidly decreasing functions will be denoted by S_n.

The topology of S_n is defined as follows.

Put

$$V_N=\left\{f\in S_n\,|\,\rho_N(f)<\frac{1}{N}\right\},$$

then $\{V_N|N=1,2,\cdots\}$ is a local base of S_n. It follows from our definition that

$1°$. S_n is a Frechet space ([10]).

$2°$. $\mathscr{D}(R^n)\subset S_n$ as sets.

Proposition 3.3. (a) *If* $f\in S_n$, then for all $K\ge 0$ and multi-index α

$$\lim_{|x|\to\infty}|x|^K|(D_\alpha f)(x)|\longrightarrow 0.$$

That is, if $f\in S_n$, *then* f *vanishes at infinity.*

(b) *If* $f\in S_n$, *then* $f\in L^1(R^n)$ *and* $f\in L^2(R^n)$.

(c) $\mathcal{D}(R^n)$ *is dense in* S_n.

Proof. (a) For all $x \in R^n$, since $|x| < (1+|x|^2)$ it is clear that

$$\lim_{|x|\to\infty} |x|^K |(D_\alpha f)(x)| \longrightarrow 0$$

by Definition 3.2.. In consequence, if we put

$C_o(R^n) = \{f : R^n \to R \mid f$ is continuous and vanishes at infinity$\}$, then $S_n \subset C_o(R^n)$

(b) We have to show that

$$f \in S_n \Longrightarrow \int_{R^n} |f(x)|\, dm_n(x) < \infty \text{ and}$$
$$\int_{R^n} |f(x)|^2\, dm_n(x) < \infty.$$

For the positive integer $N \geq n+1$, it is obvious that there is a constant M such that

$$(1+|x|)^N |f(x)| \leq M$$

for all $x \in R^n$.

Thus

$$\int_{R^n} |f(x)|\, dm_n(x) \leq (2\pi)^{-\frac{n}{2}} \cdot M \int_{R^n} (1+|x|^2)^{-N}\, dx$$
$$= (2\pi)^{-\frac{n}{2}} \cdot M \cdot 2^n \left\{ \int_o^\infty \cdots \int_o^\infty (1+x_1^2+\cdots+x_n^2)^{-N} dx_1 \cdots dx_n \right\} < \infty,$$

since

$$\int_o^\infty (1+x_1^2+\cdots+x_n^2)^{-N}\, dx_1$$
$$= \int_o^1 (1+x_1^2+\cdots+x_n^2)^{-N}\, dx_1 + \int_1^\infty (1+x_1^2+\cdots+x_n^2)^{-N}\, dx_1$$
$$\leq (1+x_2^2+\cdots+x_n^2)^{-N} + \int_1^\infty (1+x_1^2+\cdots+x_n^2)^{-N} x_1\, dx_1$$
$$\leq (1+x_2^2+\cdots+x_n^2)^{-N} + \int_o^\infty (1+x_1^2+\cdots+x_n^2)^{-N} x_1\, dx_1.$$

Moreover, by putting $x_1^2+\cdots+x_n^2 = T$, then

$$\int_o^\infty (1+x_1^2+\cdots+x_n^2)^{-N} x_1\, dx_1 = \int_{x_2^2+\cdots+x_n^2}^\infty (1+T)^{-N} \frac{1}{2}\, dT$$
$$= \frac{1}{2(N-1)} (1+x_2^2+\cdots+x_n^2)^{-N+1}.$$

Thus

$$\int_o^\infty \int_o^\infty (1+x_1^2+\cdots+x_n^2)^{-N}\, dx_1 \cdot dx_2$$
$$\leq \int_o^\infty (1+x_2^2+\cdots+x_n^2)^{-N}\, dx_2 + \frac{1}{2(N-1)} \int_o^\infty (1+x_1^2+\cdots+x_n^2)^{-N+1} dx_2.$$

Since

$$\int_0^\infty (1+x^2)^{-m}dx<\infty \text{ if } m\geq 1,$$

it follows that

$$\int_{R^n}|f(x)|dm_n(x)<\infty.$$

This implies that $S_n\subset L^1(R^n)$.

It is easy to prove that $S_n\subset L^2(R^n)$, as

$$|f(x)|^2=M^2(1+|x|^2)^{-2N}$$

implies that

$$\int_{R^n}|f(x)|^2dx\leq M^2\int_{R^n}(1+|x|^2)^{-2N}<\infty$$

if $2N\geqq n+1$.

(c) Take $f\in S_n$ and $\Psi\in\mathscr{D}(R^n)$ such that $\Psi=1$ on the unit ball of R^n. Put

$$f_\gamma(x)=f(x)\Psi(\gamma x) \quad (\gamma>0)$$

for all $x\in R^n$. Note that $f_\gamma\in\mathscr{D}(R^n)$.

For a multi-index α with $|\alpha|\leq N$ and for all $x\in R^n$,

$$\begin{aligned}&(1+|x|^2)^N D^\alpha(f-f_\gamma)\ (x)\\ &=(1+|x|^2)^N \sum_{\beta\leq\alpha}C_{\alpha\beta}(D^{\alpha-\beta}f)(x)\gamma^{|\beta|}D^\beta(1-\Psi(\gamma x)).\cdots\cdots(\text{B})\end{aligned}$$

It $|x|\leq\frac{1}{\gamma}$ the $1-\Psi(\gamma x)=0$, and thus for $|x|\leq\frac{1}{\gamma}$,

$$D^\beta(1-\Psi(\gamma x))=0$$

for every multi-index β.

Since $|x|^k\ |D^\alpha f(x)|\longrightarrow 0$, for all $x\in R^n$ as $|x|\to\infty$, it follows that

$$(1+|x|^2)^N D^{\alpha-\beta}f(x)$$

vanishes at infinity for all $\alpha\geq\beta$.

Thus, when $\gamma\longrightarrow 0$ the sum (B) above tends to zero uniformly on R.

Therefore, in the topology of S_n, $f_\gamma\to f$. ///

Recall that for each $f\in S_n$

$$\rho^N(f)=\sup_{|\alpha|\leq N}\sup_{x\in R^n}\{(1+|x|^2)^N|(D_\alpha f)(x)|\}$$

and for $f, g \in S_n$

$$d(f,g)=\sum_{N=1}^{\infty} \frac{2^{-N}\rho_N(f-g)}{1+\rho_N(f-g)},$$

which is the translation invariant metric between f and g.
We shall define the mesh of $f \in S_n$ such that

$$|f|_{S_n}=\sum_{N=1}^{\infty} \frac{2^{-N}\rho_N(f)}{1+\rho_N(f)},$$

which is compatiable with the topology of S_n.
In particular, we have to note that for $f, g \in S_n$

$$\rho_N(f-g)=0 \text{ for } N=1,2\cdots \Longrightarrow |f|_{S_n}=|g|_{S_n}.$$

Return to the Fréchet space S_n.
By Proposition 3.3, since $S_n \subset L^1(R^n)$ for each $f \in S_n$ the Fourier transform $\hat{f}$ of f is defined.
In particular, the following have already proved ([10]).

Property 1. For $f \in S_n$

$$\hat{f}(x)=\int_{R^n} f(y)e^{-ix,y}\, dm_n(y),$$

$$f(x)=\int_{R^n} \hat{f}(y)e^{ix,y}\, dm_n(y)$$

Property 2. In the space S_n

$$\begin{array}{cccc} \Lambda: & S_n & \longrightarrow & S_n \\ & \Cup & & \Cup \\ & f & \longrightarrow & \hat{f} \end{array}$$

is continuous, one to one and onto.

Theorm 3.4. *The topology of S_n can not be induced by any translation invariant metric topology* ([15]) *which turns the Fourier transform (see Property 2) into an isometry of S_n onto S_n.*

Proof. Suppose that there exists such a translation invariant metric topology T on S_n, and put

$$|f|_T = \text{the mesh of } f \in S_n \text{ by the topology } T.$$

If the Theorem 3.4 is true, then the following holds;

$$|f|_T=|\hat{f}|_T \Longleftrightarrow |f|_{S_n}=|\hat{f}|_{S_n},$$

for each $f\in S_n$.

But, as in the following, $|f|_{S_n}\neq|\hat{f}|_{S_n}$ in general.

If we take $f(x)=e^{-\frac{3}{2}|x|^2}$, then we see that $f\in S_n$. Suppose $n=1$, then $f(x)=e^{-\frac{3}{2}x^2}$ satisfies the following differential equation;

$$iDf+3xf=0.$$

So $(D^\alpha f)\hat{}=x^\alpha\hat{f}x$ and $(y^\alpha\phi)\hat{}=(-1)^{|\alpha|}D^\alpha\hat{f}(x)$ give

$$3iD\hat{f}+x\hat{f}=0.$$

Therefore $\hat{f}(\xi)=Ce^{-\frac{\xi^2}{6}}$. To determine C we have

$$C=\hat{f}(0)=\int e^{-\frac{3}{2}x^2}\,dm(x)=\frac{1}{\sqrt{3}}.$$

Finally,

$$\hat{f}(x)=\prod_{j=1}^{n}\int e^{-ix_jy_j}e^{-\frac{3}{2}y_j^2}\,dm(y_j)$$

$$=\prod_{j=1}^{n}\hat{f}(x_j)$$

Therefore this means that $\hat{f}(\xi)=3^{-\frac{n}{2}}e^{-\frac{\xi^2}{6}}=3^{-\frac{n}{2}}[f(\xi)]^{\frac{1}{9}}$. ///

As in the following example, there is an invariant metric topology on S_n which turns the Fourier transfrom into an isometry of S_n onto S_n.

Of course, this translation invariant metric topology dose not coincide with the topology of S_n.

Example 3.5. Recall that S_n is dense in $L^2(R_n)$ (Proposition 3.3.). By Property I

$$f\in S_n \Longrightarrow f(x)=\int_{R^n}\hat{f}(y)e^{ix\cdot y}\,dm_n(y)$$

Thus, for $f,g\in S_n$

$$\int_{R^n}f(x)\bar{g}(x)\,dm_n(x)=\int_{R^n}\bar{g}(x)\,dm_n(x)\cdot\int_{R^n}\hat{f}(t)e^{it\cdot x}\,dm_n(t)$$

$$=\int_{R^n}\hat{f}(t)\,dm_n(t)\cdot\int_{R^n}\bar{g}(x)e^{it\cdot x}\,dm_n(x)$$

$$=\int_{R^n}\hat{f}(t)\bar{\hat{g}}(t)\,dm_n(t).$$

That is,

$$\int_{R^n} f\bar{g}\, dm_n = \int_{R^n} \hat{f}\bar{\hat{g}}\, dm_n$$

where $\bar{g}$ is a complex conjugate function of g.

Hence we have

$$\|f\|_2 = \|\hat{f}\|_2 \quad (f \in S_n).$$

4. Fourier Transforms of S'_n

Let u be a function on R^n.

For a fixed point $x \in R^n$ we defined the following;

$$(\tau_x u)(y) = u(y-x),\ \check{u}(y) = u(-y) \quad (y \in R^n).$$

Moreover, for $u \in \mathscr{D}'(R^n)$ and $\phi \in \mathscr{D}(R^n)$ it is natural to define

$$(u*\phi)(x) = u(\tau_x \check{\phi}) \quad (x \in R^n).$$

That is, $u*\phi$ is a function on R^n. Moreover, $u*\phi \in C^\infty(R^n)$ ([10]).

Definition 4.1. An *approximate identity* on R^n is a sequence of functions h_ε of the form

$$h_\varepsilon(x) = \varepsilon^{-n} h\left(\frac{x}{\varepsilon}\right),$$

where $h \in \mathscr{D}(R^n)$, $h \geq 0$ and $\int_{R^n} h(x)\, dm_n(x) = 1$.

We have the following properties ([10]):

Property 3. (a) $\lim_{\varepsilon \to 0} \phi * h_\varepsilon = \lim_{\varepsilon \to 0} h_\varepsilon * \phi = \phi$ for each $\phi \in \mathscr{D}(R^n)$

(b) For each $u \in \mathscr{D}'(R^n)$

$$\lim_{\varepsilon \to 0} u * h_\varepsilon = \lim_{\varepsilon \to 0} h_\varepsilon * u = u.$$

Consider the Fréchet spaces $\mathscr{D}(R^n)$ and S_n. It way proved that the identity map i: $\mathscr{D}(R^n) \longrightarrow S_n$ is a continuous function ([10]). Therefore, if L is a continuous linear functional on S_n, then

$$u_L = L \circ i : \mathscr{D}(R^n) \longrightarrow R$$

is a continuous liner functional, i.e., $u_L \in \mathscr{D}'(R^n)$.

By the denseness of $\mathscr{D}(R^n)$ in S_n((c) of Proposition 3.3) shows that two distinct L's cannot give rise to the same u.

In particular, u_L can be extended to S_n and this extension of u_L is a continuous linear functional on S_n.

Definition 4.2. A *tempered distribution* is a distribution $u \in \mathscr{D}'(R^n)$, which has the continuous extension to S_n([1], [6], [7]).

Let S_n' be the dual space of S_n with the weak*-topology induced by S_n.
If we put

$$\mathscr{D}'(R^n)_T = \{u \in \mathscr{D}'(R^n) \mid u \text{ is a tempered distribution}\},$$

then as the descriptions above, we have an isomorphism

$$\mathscr{D}'(R^n)_T \cong S_n'$$

as vector spaces.

Definition 4.3. For $u \in S_n'$, we define $\hat{u}(\phi) = u(\hat{\phi})$ $(\phi \in S_n)$.
Then, by Property 2, $\phi \longrightarrow \hat{\phi}$ is continuous, linear and onto, and thus $\hat{u} \in S_n'$, since u is continuous and linear.

Moreover, for $u \in S_n'$ and $\phi \in S_n$, we define

$$(u*\phi)(x) = u(\tau_x \breve{\phi}) \quad (x \in R).$$

Then, since $\tau_x \phi \in S_n$, the above definition makes sense.

Lemma 4.4 *With the above notations, the following hold:*

(a) *The Fourier transform*

$$\Lambda: \begin{array}{ccc} S_n' & \longrightarrow & S_n' \\ \Downarrow & & \Downarrow \\ u & \longrightarrow & \hat{u} \end{array}$$

is a continuous iinear, one-to-one and onto mapping.

(b) *For $u \in S_n'$ and $\phi \in S_n$, $u*\phi$ is a tempered distribution.*

(c) *With notation in* (b), $(u*\phi)^\wedge = \hat{\phi}\hat{u}$.

Proof. (a) Since the continuous, one-to-one and onto mapping

$$\begin{array}{cccc} \Psi: & S_n & \longrightarrow & S_n \\ & \Cup & & \Cup \\ & \phi & \longrightarrow & \hat{\phi} \end{array}$$

has period 4([10]), our definition 4.3:

$$\hat{u}(\phi)=u(\hat{\phi}) \quad (\text{for all } \phi\in S_n, \text{ for all } u\in S_n')$$

shows that the mapping

$$\begin{array}{cccc} \Phi: & S_n' & \longrightarrow & S_n' \\ & \Cup & & \Cup \\ & u & \longrightarrow & \hat{u} \end{array}$$

has period 4.

Hence Φ is one-to-one, onto and linear.

The continuity of Φ is proved as follows.

Let W be a neighborhood of 0 in S_n'.

By the weak*-topology of S_n' there exist functions $\phi_1, \cdots, \phi_n\in S_n$ such that

$$\{u\in S_n' \mid |u(\phi_i)|<1 \text{ for } 1\leq i\leq m\} \subset W.$$

Define

$$V=\{u\in S_n' \mid |u(\hat{\phi}_i)|<1 \text{ for } 1\leq i\leq m\}.$$

Then V is a neighborhood of 0 in S_n', and since

$$\hat{u}(\phi)=u(\hat{\phi}) \quad (\phi\in S_n,\ u\in S_n'),$$

it follows that $\hat{u}\in W$ whenever $u\in V$.

This implies that Φ is continuous.

(b) Recall that for $f\in S_n$, $\rho_N(f)$ is the norm of $f, N=1,2,\cdots$.

Since, for $x, y\in R^n$

$$1+|x+y|^2\leq 2\ (1+|x|^2)\ (1+|y|^2),$$

it is clear that

$$\rho_N(\tau_x f)\leq 2^N(1+|x|^2)^N \rho_N(f)$$

for $x\in R^n$ and $f\in S_n$.

Since, $u\in S_n'$ is continuous on S_n and the norms ρ_N determine the topology of S_n, for each $f \in S_n$, there exist N and $C < \infty$ such that

$$|u(f)| \leq C\rho_N(f)$$

([10]). Therefore

$$|(u*\phi)(x)| = |u(\tau_x \check{\phi})| \leq 2^N C\rho_N(\phi)(1+|x|^2)^N$$

for $x \in R^n$.

Hence

$$\left| \int_{R^n} C\rho_N(\phi)(1+|x|^2)^N f(x)\, dm_n(x) \right| < \infty.$$

Therefore, since $\mathscr{D}(R^n) \subset S_n$,

$$\begin{array}{rcl} u*\phi:\ \mathscr{D}(R^n) & \longrightarrow & R \\ \Uparrow & & \Uparrow \\ f & \longmapsto & \int_{R^n} (u*\phi)(x) f(x)\, dx \end{array}$$

is a tempered distribution.

Thus $u*\phi \in S_n'$.

(c) Since $u*\phi \in S_n'$, $u*\phi$ has a Fourier transform, in S_n'.

If $\Psi \in \mathscr{D}(R^n)$ with support K,

then

$$\begin{aligned} (u*\phi)^\wedge(\hat{\Psi}) &= (u*\phi)(\check{\Psi}) = \int_{R^n} (u*\rho)(x)\Psi(-x)\, dm_n(x) \\ &= \int_{R^n} u(\tau_x \check{\phi})\Psi(-x)\, dm_n(x) \\ &= u((\phi*\Psi)^\vee) \quad \text{(see p. 157 of [10])} \\ &= \hat{u}((\phi*\Psi)^\wedge) = \hat{u}(\hat{\phi}\hat{\Psi}) = (\hat{\phi}\hat{u})(\hat{\Psi}) \cdots\cdots \text{(C)} \quad \text{([11])}. \end{aligned}$$

By (C) of Proposition 3.3., the Fourier transforms of $\mathscr{D}(R^n)$ are also dense in S_n.

Thus, the above (C) holds for every $\Psi \in S_n$.

Hence the distribution $(u*\phi)^\wedge$ and $\hat{\phi}\hat{u}$ are equal.

Theorem 4.5. *Let u be a tempered distributions on R, with compact support K such that the Fourier transform $\hat{u}$ of u is a bounded function on R^n.*

If $n=1, 2$, then $\Phi u = 0$ for every $\Phi \in C^\infty(R^n)$ which vanishes on K.

Proof. Let $\{h_t\}$ be an approximate identity on R^n such that $\int_{R^n} h(x)\, dm_n(x) = 1$ as in Definition 4.1.

Then, by (C) of Lemma 4.4

$$\|(u*h_\varepsilon)^\wedge\|_2=\|\hat{h}_\varepsilon u\|_2=\left\{\int_{R^n}|\hat{h}_\varepsilon|^2|\hat{u}|^2dm_n\right\}^{\frac{1}{2}}$$
$$\leq\|\hat{u}\|_\infty\left\{\int_{R^n}|\hat{h}_\varepsilon|^2dm_n\right\}^{\frac{1}{2}} \quad (\max_{x\in R^n}|\hat{u}(x)|=\|\hat{u}\|_\infty).$$

Since $\hat{h}_\varepsilon(x)=\hat{h}(\varepsilon x)$

$$\|(u*h_\varepsilon)^\wedge\|_2\leq\|\hat{u}\|_\infty\left\{\int_{R^n}|\hat{h}(\varepsilon x)|^2dm_n(x)\right\}^{\frac{1}{2}}$$
$$=\|\hat{u}\|_\infty\varepsilon^{-\frac{n}{2}}\left\{\int_{R^n}|\hat{h}(x)|^2dm_n(x)\right\}^{\frac{1}{2}}$$
$$=\|\hat{u}\|_\infty\cdot\|\hat{h}\|_2\cdot\varepsilon^{-\frac{n}{2}}.$$

Since $h\in\mathscr{D}(R^n)\subset S_n\subset L^2(R^n)$ by Example 3.5, $\|\hat{h}\|_2=\|h\|_2$.
Therefore $\|(u*h_\varepsilon)^\wedge\|_2=\|\hat{u}\|_\infty\|h\|_2\varepsilon^{-\frac{n}{2}}<\infty$.
Thus

$$(u*h_\varepsilon)^\wedge\in L^2(R^n),$$

and $\|u*h_\varepsilon\|_2=\|(u*h_\varepsilon)^\wedge\|_2$ by Example 3.5..
Therefore, we have

$$\|u*h_\varepsilon\|_2\leq\|\hat{u}\|_\infty\|h\|_2\varepsilon^{-\frac{n}{2}}.$$

On the other hand, since for $\phi\in\mathscr{D}(R^n)$

$$u(\phi)=(u*\hat{\phi})(0)=\lim_{\varepsilon\to0}(u*(h_\varepsilon*\check{\phi}))(0)$$
$$=\lim_{\varepsilon\to0}((u*h_\varepsilon)*\check{\phi})(0)$$
$$=\lim_{\varepsilon\to0}(u*h_\varepsilon)(\phi).$$

we have

$$|u(\phi)|=\lim_{\varepsilon\to0}\left|\int_{R^n}(u*h_\varepsilon)(x)\phi(x)\,dm_n(x)\right|$$
$$\leq\lim_{\varepsilon\to0}\int_{R^n}(u*h_\varepsilon)(x)|\ \ |\phi(x)|\,dm_n(x)$$
$$\leq\lim_{\varepsilon\to0}\|u*h_\varepsilon\|_2\|\phi\|_2 \quad ([7])$$
$$\leq\lim_{\varepsilon\to0}\|\hat{u}\|_\infty\|h\|_2\varepsilon^{-\frac{n}{2}}\|\phi\|_2$$
$$=\lim_{\varepsilon\to0}\|\hat{u}\|_\infty\|h\|_2\left\{\varepsilon^{-n}\int_{R^n}|\phi(x)|^2\,dm_n(x)\right\}^{\frac{1}{2}}.$$

Let $\phi \in \mathscr{D}(R^n)$ vanishes on K, and let H_ε be the set of all points outside K whose distance from K is less than $\varepsilon > 0$.

If supp $(\phi) \cap K$ is empty, then $u(\phi) = 0$ ([10]).

Now, suppose that supp $(\phi) \cap K$ is not empty. We set

$$\phi = \phi \chi_{H_\varepsilon} + \phi \chi_{(H_\varepsilon)^c},$$

where χ is the characteristic function and $(H_\varepsilon)^c$ is the complement of H_ε.

Then

$$u(\phi \chi_{(H_\varepsilon)^c}) = 0, \text{ so } u(\phi) = u(\phi \chi_{H_\varepsilon}).$$

Hence, we have

$$|u(\phi)| \leq \|\hat{u}\|_\infty \|h\|_2 \lim_{\varepsilon \to 0} \inf \left\{ \varepsilon^{-n} \int_{H_\varepsilon} |\phi(x)|^2 \, dm_n(x) \right\}^{\frac{1}{2}}.$$

Now assume $n = 2$, and let $x_0 \in \bar{H}_\varepsilon$ such that

$$|\phi(x_0)| = \sup_{x \in H_\varepsilon} |\phi(x)|.$$

Then there exists $a \in K$ such that $|x_0 - a| \leq \varepsilon$.

Put

$$(x_0 - \mathrm{a}) = (h . K),$$

then we have

$$\begin{aligned} \phi(x_0) &= \phi(a) + h \frac{\partial \phi}{\partial x}(p) + k \frac{\partial \phi}{\partial y}(p) \\ &= h \frac{\partial \phi}{\partial x}(p) + k \frac{\partial \phi}{\partial y}(p) \end{aligned}$$

for some $p \in$ line segment joining a and x_0,

and

$$\varepsilon^{-2} \int_{H_\varepsilon} |\phi(x)|^2 \, dm_2(x) = \varepsilon^{-2} \int_{H_\varepsilon} \left[\left(\frac{\partial \phi}{\partial x}(p) \right)^2 h^2 + 2 \left| \frac{\partial \phi}{\partial x}(p) \frac{\partial \phi}{\partial y}(p) hk + \left(\frac{\partial \phi}{\partial y}(p) \right)^2 k^2 \right] dm_2(x)$$

Since $\left| \frac{h}{\varepsilon} \right| \leq 1$ and $\left| \frac{k}{\varepsilon} \right| \leq 1$,

$$\varepsilon^{-2} \int_{H_\varepsilon} |\phi(x)|^2 dm_2(x) \leq \left\{ \left(\frac{\partial \phi}{\partial x}(p) \right)^2 + 2 \left| \frac{\partial \phi}{\partial x}(p) \cdot \frac{\partial \phi}{\partial y}(p) \right| + \left(\frac{\partial \phi}{\partial y}(p) \right)^2 \right\} \cdot \frac{1}{2\pi} \cdot \mathrm{Area}(H_\varepsilon),$$

where Area(H_ε) is the area of H_ε.

If $\varepsilon \to 0$ and $p \to a_0$.
then since

$$\lim\ \inf \left\{ \left(\frac{\partial \phi}{\partial x}(p)\right)^2 + 2\left|\frac{\partial \phi}{\partial x}(p)\cdot\frac{\partial \phi}{\partial y}(p)\right| + \left(\frac{\partial \phi}{\partial y}(p)\right)^2 \right\} < \infty$$

and Area $(H_\varepsilon) \longrightarrow 0$,
we get

$$\varepsilon^{-2}\int_{H_\varepsilon} |\phi(x)|^2\, dm_2(x) \longrightarrow 0.$$

Therefore it follows that $|u(\phi)| = 0$.

Obviously, it hold for $n=1$, i.e., $\varepsilon^{-1}\int_{H_\varepsilon} |\phi(x)|\, dm(x) \longrightarrow 0$.

Since $(\Psi u)(\phi) = u(\Psi\phi)$ and $\Psi\phi \in \mathscr{D}(R^n)$ which vanishes on K for every $\phi \in \mathscr{D}(R^n)$, $\Psi u = 0$. ///

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