HONAM MATHEMATICAL JOURNAL
Volume 9, Number 1. August 1987

The Automorphisms of the Unipotent Radical of Certain Parabolic Subgroups of $S_{p_{2l}}(K)$

by

Bok-Hee Im

Department of Mathematics, University of Notre Dame
Notre Dame, Indiana 46556, U.S.A.

1. Introduction

The study of Chevalley groups G of Lie type '$\mathfrak{g}$' over an arbitrary field K, usually denoted by $\mathfrak{g}(K)$, has largely been motivated by the problem of classifying all finite simple groups. Although this problem has been solved, very much is still unknown about the structure of the group itself.

One subgroup of particular importance is the parabolic subgroup P. This subgroup admits the Levi decomposition $P=L.U$, where L is the $Levi$ subgroup and U is the unipotent radical of B. If B is the Borel subgroup, then the unipotent radical of B is a maximal unipotent subgroup of $\mathfrak{g}(K)$. Gibbs [4] has characterized the automorphisms of U when $P=B$ and char$(K)\neq 2,3$, Khor [8] has done the case when P is not the Borel subgroup and $G=A_\ell(K)$.

We characterize the automorphisms of U when P is not the Borel subgroup in the case when $G=C_\ell(K)$. There are two cases to consider depending on the flag F whose stabilizer is $P=P(F)$: Let V be a 2ℓ-dimensional vector space over K and let a family $F=(V_0, V_1\cdots, V_n)$ of isotropic subspaces of V be a flag where $O=V_0\subset V_1\cdots\subset V_n$.

In this paper we only discuss the case of dim $V_n=\ell$ where the associated root system is isomorohic to a reduced root system of type C_n. For the case of dim $V_n<\ell$, the situation is analogous to a non-reduced root system of type BC_n and we need to check a lot of details which are more complicated than ones of type C_n. Therefore it is recommended for the reader to refer the author's doctoral dissertation which will be published in the near future for the latter(BC_n) case and the details of the former (C_n) case which are not explained in this paper.

Also it will be extremely helpful for the reader to study two papers, [4] and especially [8], from which we borrow notations and many results.

Received April 25, 1987. Accepted May7, 1987.

In section 2 we explain some new notations and preliminaries.

Section 3 deals mainly with abelian normal subgroups to make an automorphism σ of U invariant on each root subgroup mod U_2.

In section 4 we make our automorphism become trivial mod U_2 by studying a diagonal automorphism.

In section 5 we use inner automorphisms to further trivialize the autnmorphism σ.

Finally, in section 6 we define new extremal automorphisms and use one more inner and a central automorphism to obtain the following main theorem:

Theorem: Let σ be any automorphism of U. Then there exist diagonal(d), field (f), inner (i), extremal(e) and central(c) automorphisms such that $\sigma=d.f.i.e.c.$

2. Notations and Preliminaries

The construction of Chevalley groups of Lie type over an arbitrary field K is well known. Good sources of reference can be found in [2], [6] and [10].

Let $G=\mathrm{Sp}_{2\ell}(K)$ be a symplectic group over an arbitrary field K, char$(K)\neq 2,3$. Let P be a Parabolic subgroup of G where P is not the Borel subgroup. Let P be a stabilizer of a flag $F=(V_0, V_1,\cdots,V_n)$ such that dim $V_n=\ell$. Then in matrix notations, relative to a symplectic basis, P consists of all $2n\times 2n$ block matrices A having partitioned form,

$$\begin{pmatrix}
A_{1,1} & A_{1,2}\cdots A_{1,n} & A_{1,\bar{n}}\cdots A_{1,\bar{2}} & A_{1,\bar{1}} \\
 & A_{2,2}\cdots A_{1,n} & A_{2,\bar{n}}\cdots A_{2,\bar{2}} & A_{2,\bar{1}} \\
 & \cdots\cdots\cdots & & \\
 & A_{n,n} & A_{n,\bar{n}}\cdots A_{n,\bar{2}} & A_{n,\bar{1}} \\
 & & A_{\bar{n},\bar{n}}\cdots A_{\bar{n},\bar{2}} & A_{\bar{n},\bar{1}} \\
 & & \cdots\cdots & \\
 & & A_{\bar{2},\bar{2}} & A_{\bar{2},\bar{1}} \\
 & & & A_{\bar{1},\bar{1}}
\end{pmatrix}$$

where $A_{i,j}\in M_{m_i\times n_{j-1}}(K)$ where m_i, n_{j-1} are positive integers and $1\leq i\leq j\leq 2n$. Let us denote $\bar{i}=2n-i+1$ for the convenience and note $m_i=m_{\bar{i}}$ and $m_j=n_{j-1}$, that is, the sizes of the diagonal blocks are symmertrical about the center. Assume $n\geq 4$ throughout this paper.

So L consists of those for which $A_{i,j}=0$ whenever $i<j$, and U consists of those for

which $A_{i,i}=I(m_i\times m_i$ identity matrix). So U consists of those unimodular, upper triangular block matrices. Let $E_{i,j}$ denote the(i,j)-block of size as same as $A_{i,j}$. Then any element A of U can be written as $I+\sum_{i<j}A_{i,j}E_{i,j}$ where $A_{i,j}E_{i,j}$ denotes the matrix $A_{i,j}$ in the(i,j)-block(not matrix multiplication!). However G consists of all matrices A over K such that ${}^{T}\boldsymbol{A}\boldsymbol{J}\boldsymbol{A}=\boldsymbol{J}$,

$$\text{where } \boldsymbol{J}=\begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \text{ and } J=\begin{pmatrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0 \end{pmatrix}. \text{ Consider } \boldsymbol{J}=\sum_{k=1}^{2n} J_{k,\bar{k}}E_{k,\bar{k}}.$$

Then $J_{k,\bar{k}}=-J_{\bar{k},k}$ so we often write J for $J_{k,\bar{k}}$ when $i=1,\cdots,n$.

Therefore any element A of U should satisfy the following equation unless $i=k=j$:

$$\sum_{1\le i\le k\le j\le 2n} {}^{T}A_{k,i}\,J_{k,\bar{k}}\,A_{\bar{k},\bar{j}}=0\cdots(*)$$

From the above(*), U contains any element of the form

$$I+A_{i,\bar{i}}E_{i,\bar{i}} \text{ wher } A_{i,\bar{i}}=J^{T}(A_{i,\bar{i}})\,J \text{ for } 1\le i\le n$$

$$\text{and } I+A_{i,j}E_{i,j}+A_{\bar{j},\bar{i}}E_{\bar{j},\bar{i}} \text{ for } 1\le i<j\le 2n=\bar{1},\ \bar{i}\ne j,$$

$$\text{where } A_{\bar{j},\bar{i}}=-J_{i,\bar{j}}{}^{T}(A_{i,j})\,J_{i,\bar{i}} \text{ so } A_{\bar{j},\bar{i}}=-J^{T}(A_{i,j})J$$

$$A_{i,\bar{j}}=J^{T}(A_{j,\bar{i}})\,J \text{ for } j\le n.$$

Nataion: Denote $\bar{A}=J^{T}AJ$. That is to say, get $\bar{A}$ by transposing A about the second(not\\but/) diagonal. So $A=\bar{A}$ if and only if A is symmetric about the second diagonal. Note $\overline{AB}=\bar{B}\bar{A}$. Then we set for $i=1,2,\cdots,n-1$,

$X_{R_i}(A_{i,i+1})=I+A_{i,i+1}E_{i,i+1}+A_{\overline{i+1},\bar{i}}E_{\overline{i+1},\bar{i}}$ where $A_{\overline{i+1},\bar{i}}=-\overline{A_{i,i+1}}$

and $X_{R_n}(A_{n,\bar{n}})=I+A_{n,\bar{n}}E_{n,\bar{n}}$ where $A_{n,\bar{n}}=\overline{A_{n,\bar{n}}}$

Let $R_{i,\bar{i}}=2\sum_{i\le k<n}R_k+R_n$ for $1\le i\le n$ and

$R_{i,j}=\sum_{i\le k<j}R_k$ and $R_{i,\bar{j}}=R_{i,j}+R_{j,\bar{j}}$ for $1\le i<j\le n$.

Then the set $\pi=\{R_i\}_{1\le i\le n}$ generates a root system Φ isomorphic to C_n with the highest root $R_N=R_{1,\bar{1}}=2(R_1+\cdots+R_{n-1})+R_n$. Thus we have an analogous notion of the height of a root in the positive root system Φ^{+} and $\mathrm{ht}(R_N)=h=2n-1$ where Φ^{+} is the set of all the above R_{ij}, $R_{i\bar{j}}$, $R_{i\bar{i}}$ for $1\le i<j\le n$.

Also we can associate each root in Φ^{+} with a root subgroup as follows:

$X_{R_{i,\bar{i}}}=X_{i,\bar{i}}=\{x_{i,\bar{i}}(A): A=A_{i,\bar{i}}\in M_{i,\bar{i}}(K)$ where

$X_{i\bar{i}}(A)=I+AE_{i\bar{i}}$ with $A=\bar{A}\}$ for $1\le i\le n$.

$X_{R_{i,j}}=X_{i,j}=\{X_{ij}(A): A\in M_{ij}(K)$ is any $m_i\times m_j$

matrix where $X_{i,j}(A)=I+AE_{i,j}-\bar{A}E_{\bar{j},\bar{i}}\}$ and similarly

$X_{R_{i,\bar{j}}} = X_{i,\bar{j}} = \{X_{i,\bar{j}}(A) = I + AE_{i,\bar{j}} + \bar{A}E_{j,\bar{i}}$ for any $A \in M_{ij}(K)\}$ for $1 \le i < j \le n$.

Then we are equipped with the following commutator relations:
$[a,b] = a^{-1}b^{-1}ab$ and so $[a,b]^{-1} = [b,a]$ for any $a, b \in U$. If $R+S \notin \Phi^+$, then $[X_R, X_S] = 1$. If $R+S \in \Phi^+$, then for $1 \le i < j \le n$, and for $1 \le k \le t \le n$ the products can be written as,

$$
\begin{aligned}
[X_{ij}(A),\ X_{kt}(B)] &= X_{i,t}(AB) && \text{if } j=k,\\
[X_{ij}(A),\ X_{k,\bar{t}}(B)] &= X_{i,\bar{t}}(AB)\ X_{i,\bar{t}}(-AB\bar{A}) && \text{if } j=k=t,\\
&= X_{i,\bar{t}}(AB) && \text{if } j=k\neq t,\\
&= X_{i,\bar{t}}(A\bar{B}+B\bar{A}) && \text{if } i=k, \text{ and } j=t,\\
&= X_{i,\bar{t}}(A\bar{B}) && \text{if } i<k, \text{ and } j=t,\\
&= X_{k,\bar{t}}(B\bar{A}) && \text{if } i>k, \text{ and } j=t,
\end{aligned}
$$

and $X_R(A)\,X_R(B) = X_R(A+B)$ for all $R \in \Phi^+$.

Now we claim that U is generated by all root subgroups X_R for $R \in \Phi^+$. Then by the above relations every element of U can be expressed uniquely in the form

$$X_{R_1}(A_{12}).\ X_{R_2}(A_{23}) \cdots X_{R_N}(A_{1,\bar{1}})$$

where $\{R_i\} \in \Phi^+$ are arranged in increasing order.

For $1 \le k < 2n$, let U_k denote the subgroup of U generated by all root subgroups of U corresponding to the roots of height at least k. Then

$$U_k = \{\prod_{R \in \Phi^+} X_R(A) = 0 \text{ whenever } \mathrm{ht}(R) < k \text{ for } A \in M_R(K)\}.$$

By the commutator relations, we obtain,

Proposition 2.1 U is generated by the fundamental root subgroups X_R for all $R = R_i \in \pi$.

Note: X_R will be also written as X_i for $R = R_i \in \pi$.

Proposition 2.2 $U = U_1 \supset \cap_2 \supset \cdots U_{2n} = 1$ is the lower central series of U.

3. The Abelian Normal Subgroups of U

For $1 \le s \le n$, let $\mathscr{M}_s$ be the subgroup of U generated by all the root subgroups X_{uv} for $1 \le u \le s < v \le 2n$. Then $\mathscr{M}_s$ is a normal subgroup of U. Indeed for $i = 1, 2, \cdots, n$, $[X_i; X_{uv}] \subset \mathscr{M}_s$ by the commutator relations. Also since $[\mathscr{M}_s; \mathscr{M}_s: \mathscr{M}_s] = 1$, $\mathscr{M}_s$ is a nilpotent subgroup of class 2 for $i = 1, 2, \cdots, n-1$. Especially $\mathscr{M}_s$ is an abelian normal subgroup because all the commutator products are trivial. Moreover we have the following desirable proposition:

Proposition 3.1 $\mathscr{M}_n$ is the unique maximal abelian normal subgroup of U.

Proof: Let $\mathscr{N}$ be any normal subgroup of U such that $\mathscr{N}\supsetneqq\mathscr{M}_n$. Then it is enough to show that $\mathscr{N}$ is not abelian.

Take any element A such that $A\in\mathscr{N}/\mathscr{M}_n$. Then A can be expressed uniquely (arranged in increasing order) in the form $\Pi_{i<j}X_{ij}(A_{ij})$ mod $\mathscr{M}_n$, where $A_{ij}\neq 0$ for some $A_{ij}\in M_{ij}(K)$ and $A\in U_{j-i}$ mod $\mathscr{M}_n$ for $1\leq i<j\leq n$.

Chose i,j such that j-i is minimal satisfying $A_{ij}\neq 0$

We claim $\mathscr{N}$ always has an element $B=X_{1n}(B)$ mod $\mathscr{M}_n$ for some nonzero $B\in M_{1n}(K)$. There are four cases to check by using the normality of $\mathscr{N}$:

1) For $1<i<j<n$, we have

$$[[X_{1i}(V),\ A],\ X_{jn}(W)]=X_{1n}(VA_{ij}W)\in\mathscr{N} \text{ mod } \mathscr{M}_n$$

for all $V\in M_{1i}(K)$, $W\in M_{jn}(K)$.

Then there exists a nonzero $B=VA_{ij}W\in M_{1n}(K)$ for some $V\in M_{1i}(K)$ and $W\in M_{jn}(K)$ since $A_{ij}\neq 0$.

2) Suppose $i=1$ and $j\neq n$. Then we have

$[A,\ X_{jn}(W)]=X_{1n}(A_{1j}W)\in\mathscr{N}$ mod $\mathscr{M}_n$ for all $W\in M_{jn}(K)$. Then there exists a nonzero $B=A_{ij}W\in M_{1n}(K)$ for some $W\in M_{jn}(K)$.

3) Suppose $i\neq 1$ and $j=n$. Then $[X_{1i}(V),\ A]=X_{1n}(VA_{in})\in\mathscr{N}$ mod $\mathscr{M}_n$ for all $V\in M_{1i}(K)$. So there exists a nonzero $B=VA_{in}\in M_{1n}(K)$ for some $V\in M_{1i}(K)$.

4) Lastly suppose $i=1$ and $j=n$. Then $B=A_{ij}$.

Now Let us prove that $\mathscr{N}$ is not abelian. In fact, we have the following element contained in $[\mathscr{N};\mathscr{N}]$:

$$\begin{aligned}[B;[B;X_n(W)]]&=[B;\ X_{1n}(BW)\ X_N(-BW\bar{B})]=[X_{1n}(B);\ X_{1n}(BW)]\\&=X_N(B\overline{BW}+BW\bar{B})=X_N(2BW\bar{B})\end{aligned}$$

which is not the identity because $2BW\bar{B}\neq 0$ for some $W\in M_{n,\bar{n}}(K)=\{W:W=\bar{W}\}$ since char $K\neq 2$. ///

Therefore we have the following useful corollary:

Corollary 3.2 σ induces an antomorphism on the quotient group $U/\mathscr{M}_n$ which is ismorphic to Khor's unipotent group of type A_{n-1} [8].

Therefore we can apply his results to get the following:

For $i=1,2,\cdots,n-1$, either (1) $\sigma(X_i)\subset X_i$ mod $U_2\mathscr{M}_n$ or

(2) $\sigma(X_i)\in X_{n-i}$ uod $U_2\mathscr{M}_n$.

But we can rule the second possibility out by considering $[X_1, \mathscr{M}_n]$ and $[X_{n-1}, \mathscr{M}_n]$ mod U_3 and furthermore we get the following proposition:

Proposition 3.3 Let σ be an automorphism of U. Then σ is invariant on each fundamental root subgroup X_i mod U_2, i.e., $\sigma(X_i) \subset X_i$ mod U_2.

Idea of Proof: Since $\mathscr{M}_S$ is a nilpotent subgroup of class 2 for $i \leq n-1$, σ should send $\mathscr{M}_S$ to a nilpotent subgroup of class 2. Proposition 2.2 about the classification of abelian normal subgroups and the proof of proposition 6.1 from Khor [8] are needed. A couple of pages of manipulating delicate commutator calculations are necessary. ///

4. The Diagonal Automorphism of U

For the definitions and some properties of "elementary" automorphisms which are necessary for the characterization of the automorphisms of U, the reader may refer to [2], [4] and [8].

Assume proposition 6.9 and its proof from Khor [8]. Also assume the above proposition 3.3. Then $\sigma: X_i(A_i) \longrightarrow X_i(\sigma_i(A_i))$ mod U_2 where σ_i is additive, since

$$\begin{aligned} X_i(\sigma_i(A_i+B_i)) &= \sigma(X_i(A_i+B_i)) = \sigma(X_i(A_i)X_i(B_i)) \\ &= \sigma(X_i(A_i))\,\sigma(X_i(B_i)) = X_i(\sigma_i(A_i))X_i(\sigma_i(B_i)) \\ &= X_i(\sigma_i(A_i)+\sigma_i(B_i)). \end{aligned}$$

We have $[X_{n-1}(A_{n-1}), X_n(A_n)] = 1$ mod U_3 if and only if $A_{n-1}A_n = 0$ if and only if

$\sigma_{n-1}(A_{n-1})\sigma_n(A_n) = 0$ if and only if $P_{n-1}(A_{n-1})^{\theta} Q_{n-1}\,\sigma_n(A_n) = 0$

if and only if $(A_{n-1})^{\theta} Q_{n-1}\sigma_n(A_n).c^2.\bar{Q}_{n-1} = 0$ where $c = c_1.c_2 \cdots c_{n-2}$.

Put $\tau(A_n) = Q_{n-1}\sigma_n(A_n).c^2.\bar{Q}_{n-1}$. Then we claim that $\tau(A_n) = k(A_n)^{\theta}$ for some nonzero $k \in K$. Therefore $\sigma_n(A_n) = kD_n(A_n)^{\theta}\bar{D}_n$.

Now define a diagonal automorphism d of U by $d(A) = DAD^{-1}$ where D is the diagonal block matrix$(D_1, D_2, \cdots, D_n, k^{-1}\bar{D}_n^{-1}, \cdots, k^{-1}\bar{D}_2^{-1}, k^{-1}\bar{D}_1^{-1})$ and the above discussion can be formulated as,

Proposislon 4.1 If σ is invariant on X_i's mod U_2, then there is a diagonal (d) and a field(f) automorphism such that $f^{-1}d^{-1}\sigma$ is trivial on U mod U_2.

5. The Inner Automorpbisms of U

We introduce a lexicgoraphical ordering of the positive roots in Φ^+ as follows;

$R_{i,j}<R_{k,t}$ if and only if either $i<k$ or $i=k$, $j<t$.

Lemma 5.1 Let $R_i\in\pi$ be a fundamental root, $S\in\Phi$, $2\leq ht(S)\leq h-1$. If $S-R_i$ is not a root, then either

1) $S+R_j\in\Phi^+$ and $R_i+R_j\notin\Phi^+$ for some $R_j\in\pi$, $R_j\neq R_i$ or

2) $S+R_i+R_j\in\Phi^+$ and $2R_i+R_j\notin\Phi^+$ for some $R_j\in\pi$, $R_j\neq R_i$ or

3) $S+2R_i+R_j\in\Phi^+$ and $2R_i+R_j\in\Phi^+$ for some $R_j\in\pi$, $R_j\neq R_i$ or

4) $S+2R_i$ is the highest roo.

Proof: See lemma 6.6 from Gibbs [4].

We only state the following two lemmas without proof.

Lemma 5.2 Let σ be an automorphism of U which is trivial on U mod U_m, for $2\leq m\leq h-1$. Suppose $R_i\in\pi$ and let

$$\sigma:\ X_{R_i}(A_i)\longrightarrow X_{R_i}(A_i)\,X_S(f(A_i))\cdots$$

where $ht(S)=m$, $f\neq 0$. Then $S-R_i\in\Phi^+$ except in the case (4) of lemma 5.1.

Lemma 5.3 Let σ be an automorphism of U which is trivial on U mod U_m where $ht(S)=m-1\leq h-3$. If σ: $X_{R_i}(A_i)\longrightarrow X_{R_i}(A_i)\,X_{S+R_i}(f(A_i))\cdots$, then one of the following holds:

(Ⅰ) If there exists $R_j(\neq R_i)$ such that $S+R_j\in\Phi^+$ and

σ: $X_{R_j}(A_j)\longrightarrow X_{R_j}(A_j)X_{S+R_j}(g(A_j))\cdots$, then either (1) or (2):

(1) $R_i+R_j\notin\Phi^+$: If $R_i<R_j$, then there is a $T\in M_S(K)$ such that $f(A_i)=A_iT$ for all $A_i\in M_{R_i}(K)$ and $g(A_j)=-TA_j$ or $T\bar{A}_j$ depending on $S\in\Phi^+$ for all $A_j\in M_{R_j}(K)$.

If $R_i>R_j$, then there is a $T\in M_S(K)$ such that $g(A_j)=A_jT$ for all $A_j\in M_{R_j}(K)$ and $f(A_i)=-TA_i$ or $T\bar{A}_i$ depending on $S\in\Phi^+$ for all $A_i\in M_{R_i}(K)$.

(2) $R_i+R_j\in\Phi^+$: $R_i<R_j$, then there is a $T\in M_S(K)$ such that $f(A_i)=A_iT$ for all $A_i\in M_{R_i}(K)$ and $g(A_j)=A_jT+T\bar{A}_j$ for all $A_j\in M_{R_j}(K)$.

If $R_i>R_j$, If then there is a $T\in M_S(K)$ such that $g(A_j)=A_jT$ for all $A_j\in M_{R_j}(K)$ and $f(A_i)=A_iT+T\bar{A}_i$ for all $A_i\in M_{R_i}(K)$.

(Ⅱ) Otherwise, that is, let R_i be the only simple root in π such that $S+R_i\in\Phi^+$, then

(3) If $R_i<S$, then there is a $T\in M_S(K)$ such that $f(A_i)=A_iT$ for all $A_i\in M_{R_i}(K)$.

(4) If $R_i>S$, then there is a $T\in M_S(K)$ such that $f(A_i)=-TA_i$ or $T\bar{A}_i$ depending on whether $S=R_{1,i}(2\leq i\leq n)$ or $S=R_{1,\overline{i+1}}(4\leq i+1\leq n)$ for all $A_i\in M_{R_i}(K)$.

Proposition 5.4 Let σ be an automorphism of U which is trivial on U mod U_2. Then there exists an inner automorphism (i) such that $i^{-1}\sigma$ acts trivially on U mod U_{h-2}.

Proof: Use the induction on k: it is enough to show that if σ acts trivially on U mod U_k for $2\leq k\leq h-3$, then there exists an inner automorphism (i) such that $i^{-1}\sigma$ acts trivially on U mod U_{k+1}. Proposition follows mainly from lemma 5.3. ///

6. Tbe Extremal Automorpbisms of U

For the extremal automorphisms we need to construct two new ones.

1) Let $f_1=f$: $M_{m_1\times n_1}(K)\longrightarrow M_{n_1\times n_1}(K)$ be an additive function such that

$$A_1f_1(A_1')=A_1'f_1(A_1) \text{ and } f_1(A_1)=\overline{f_1(A_1)}$$

for all $A_1, A_1'\in M_{R_1}(K)=M_{m_1\times n_1}(K)$. For any $A=I+\sum_{u<v}A_{u,v}E_{u,v}\in U$, define $e_f=e_1$: $U\longrightarrow U$ by $e_1(A)=T=I+\sum_{i<j}T_{i,j}E_{i,j}$ where

$$T_{1,2n}=A_{1,2n}-1/6A_{1,2}f(A_{1,2})\bar{A}_{1,2}$$
$$T_{2,2n}=A_{2,2n}-1/2f(A_{1,2})\bar{A}_{1,2}$$
$$T_{1,2n-1}=A_{1,2n-1}+1/2\,A_{1,2}f(A_{1,2})$$
$$T_{2,2n-1}=A_{2,2n-1}+f(A_{1,2}) \text{ and } T_{i,j}=A_{i,j} \text{ otherwise.}$$

$$\text{i.e. } e_1(A)=\begin{pmatrix} I & 0\cdots\cdots 0 & (1/2\,f(A)\bar{A}) & (1/3\,Af(A)\bar{A}) \\ & I\cdots\cdots 0 & f(A) & (1/2\,f(A)\bar{A}) \\ & & & 0 \\ & & & I \end{pmatrix}A \text{, where } A=A_{1,2}.$$

Then $e_1(X_{R_1}(A))=X_{R_1}(A)X_{R_2,\bar{2}}(f(A))X_{R_1,\bar{2}}(-1/2Af(A))X_{R_N}(1/3Af(A)\bar{A})$

and e_1 acts trivially on the root subgroups X_R for $R\neq R_1$. Indeed we can show that $e_1=e_f$ is an automorphism of U using matrix multiplication and properties of $f(=f_1)$;

$$e_f(AB)=e_f(A)e_f(B) \text{ and } e_f(e_{-f}(A))=A \text{ for all } A, B\in U.$$

(2) Let $f_1=f$: $M_{m_1\times n_1}(K)\longrightarrow M_{m_1\times n_1}(K)$ be an additive function such that

$$A\overline{f(A')}+f(A')\bar{A}=A'\overline{f(A)}+j(A)\bar{A}'$$

for all $A, A'\in M_{R_1}(K)=M_{m_1\times n_1}(K)$.

Hor any $A\in U$, define $e_f=e_2$: $U\longrightarrow U$ by $e_2(A)=S=I+\sum_{i<j}S_{i,j}E_{i,j}$ where

$$S_{1,2n}=A_{1,2n}+1/2\{A_{1,2}\overline{f(A_{1,2})}-f(A_{1,2})\overline{A_{1,2}}\}$$

$S_{2,2n}=A_{2,2n}+\overline{f(A_{1,2})}$

$S_{1,2n-1}=A_{1,2n-1}+f(A_{1,2})$ and $S_{i,j}=A_{i,j}$ otherwise.

$$\text{i.e. } e_2(A)=\begin{pmatrix} I & 0 & f(A) & 1/2(A\overline{f(A)}+f(A)\bar{A}) \\ & I & 0 & \overline{f(A)} \\ & & & 0 \\ & & & I \end{pmatrix} A, \text{ where } A=A_{1,2}.$$

Then $e_2(X_{R_1}(A))=X_{R_1}(A)X_{R_{2\bar{2}}}(f(A))X_{R_N}(-1/2\{A\overline{f(A)}+f(A)\bar{A}\})$

and e_2 acts trivially on the root subgroups X_R for $R\neq R_1$. We can also show that e_2 is an automorphism of U similarly as e_1.

Proposition 6.1 Let σ act trivially on U mod U_{h-2}. Then there exist an inner automorphism (i_1) and an extremal automorphism (e_1) of U such that $e_1^{-1}\, i_1^{-1}\sigma$ acts trivially on U mod U_{h-1}.

Proof: By lemma 5.2 σ acts as follows on each fundamental root subgroups X_i mod U_{h-1}:

$$X_1(A_1)\longrightarrow X_1(A_1)\, X_{1\bar{3}}(g_1(A_1))\, X_{2\bar{2}}(f_1(A_1)),$$
$$X_2(A_2)\longrightarrow X_2(A_2)\, X_{2\bar{2}}(f_2(A_2)),$$
$$X_3(A_3)\longrightarrow X_3(A_3)\, X_{1\bar{3}}(f_3(A_3)), \text{ and}$$
$$X_i(A_i)\longrightarrow X_i(A_i) \text{ for } i=4,\cdots,n$$

where for all $A_i\in M_{R_i}(K)$, f_i and g_1 are additive. Then applying lemma 5.3 as in proposition 5,4 we can find an inner automorphism (i_1) of U such that $i_1^{-1}\,\sigma$ acts trivially on X_j for all $j\neq 1$ and

$$X_1(A_1)\longrightarrow X_1(A_1)X_{2\bar{2}}(f_1(A_1)) \text{ mod } U_{h-1}.$$

Now since $[X_1(A_1); X_1(A_1')]=1$, the image of this commutator under $i_1^{-1}\sigma$ has a term in $X_{1\bar{1}}=X_{R_N-R_1}$ which is trivial, so we have $A_1.f_1(A_1')=A_1'.f_1(A_1)$ and $f_1(A_1)=\overline{f_1(A_1)}$. Thus there exists an extremal automorphism (e_1) of U such that $i_1^{-1}\sigma=e_1$ mod U_{h-1}.

Proposition 6.2 Let σ act trivially on U mod U_{h-1}. Then there exist an inner automorphism (i_2) and an extremal automorphism (e_2) of U such that $e_2^{-1}\, i_2^{-1}\sigma$ acts trivially on U mod U_h.

Proof: By lemma 5.2 again, σ acts as follows on each fundamental root subgroup X_i mod U_h:

$$X_i(A_i) \longrightarrow X_i(A_i)X_{1\bar{2}}(f_i(A_i)) \text{ for } i=1,2 \text{ and}$$
$$X_i(A_i) \longrightarrow X_i(A_i) \text{ for } i=3,\cdots,n$$

where for all $A_i \in M_{R_i}(K)$, f_i are addilive. When $i=2$, from $[X_1(A_1); X_2(A_2)]$, we get

$$A_1A_2=0 \text{ implies } A_1\overline{f_2(A_2)}+f_2(A_2)\bar{A}_1=0.$$

Then we claim there exists $T \in M_S(K)$ with $S=R_N-R_1-R_2$ such that $f_2(A_2)=T\bar{A}_2$ for all $A_2 \in M_{R_2}(K)$. Therefore the inner automorphism i_2 induced by $X_S(-T)$ maps $X_2(A_2)$ onto $X_2(A_2)X_{1\bar{2}}(f_2(A_2))$ and acts trivially on X_i for $i \neq 2$. So we have

$$i_2^{-1}\sigma: X_1(A_1) \longrightarrow X_1(A_1)X_{1\bar{2}}(f_1(A_1)) \bmod U_h \text{ and}$$
$$i_2^{-1}\sigma: X_i(A_i) \longrightarrow X_i(A_i) \bmod U_h \text{ for } i \neq 1.$$

From $[X_1(A); X_1(A')]=1$ for all $A, A' \in M_{R_1}(K)$, the image of this commutator under $i_2^{-1}\sigma$ has a term in X_N which should be trivial, so we get

$$A\overline{f(A')}+f(A')\bar{A}=A'\overline{f(A)}+f(A)\bar{A}'$$

Thus for some extremal automorphism (e_2) of U, $e_2^{-1}\, i_2^{-1}\sigma=1$ on U mod U_h. ///

Theorem 6.3 Let σ be any automorphism of U. Then there exist diagonal (d), field (f), inner (i), extremal (e) and central (c) automorphisms such that

$$\sigma=d.f.i.e.c.$$

Proof: From the above two propositions 6.1 and 6.2 if σ acts trivally on U mod U_{h-2}, then we have $e_2^{-1}.i_2^{-1}.e_1^{-1}.i_1^{-1}.\sigma$ as a central automorphism (c). But since an extremal automorphism e_1 fixes a root subgroup $X_{1\bar{3}}=X_{R_N-R_1-R_2}$, i_2 and e_1 commute and so $e_2^{-1}.e_1^{-1}.i_1^{-1}.i_2^{-1}.\sigma=e^{-1}.i^{-1}.\sigma=c$. Now theorem follows by combining propositions 3.3, 4.1, 5.4. ///

References

1. N. Bourbaki, Groupes et algebres de Lie, Chap. 7~8, Paris: Hermann, 1975.
2. R.W. Carter, Simple Groups of Lid Type, Wiley, New York, 1972.
3. R.W. Carter, Finite Groups of Lie Type, Wiley, New York, 1985.
4. J.A. Gibbs, Automorphisms of certain unipotent groups, *J. Algebra* **14**(1970), 203~228.
5. D. Gorenstein, Finite Groups, Chelsea, New York, 1968.
6. J.E. Humphereys, Introduction to Lie Algdbras and Representation Theory, Springer-Verlag, Berlin/Heidelberg/New York, 1980.
7. H.G. Jacobs, Jr., Coherence invariant mappings on Kronecker products, *Amer. J. Math.* **77**(1955), 177~189.

8. H.P. Khor, The Automorphims of the Unipotent Radical of Certain Parabolic Subgroups of $GL(\ell+1, K)$, *J. Algebra* **96**(1985), 54~77.
9. R. Steinberg, Automorphism of finite linear groups, *Canad. J. Math*, **12**(1960), 606~615.
10. R. Steinberg, Lectures on Chevally groups, Yale University, 1967.
11. W. Wakins, Linear maps that preserve commuting pairs of matrices, *Linear Algebra Appl.* **14**(1976), 29~35.
12. W.J. Wong, Maps on simple algebras preserving zero products: Ⅱ Lie algebras of linear type, *Pacific J. Math.* **92**(2) 1981, 469~488.