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A NOTE ON JOINT SPECTRA AND HOMOMORPHISMS OF BANACH ALGEBRAS

CHONG-ROCK LIM

1. Introduction

The notions of the joint spectrum and analytic functions of a family of elements in a commutative Banach algebra were fiirst introduced and studied by Aren and Caldorón ([1]). At the same time, applications of complex analysis in the theory of Banach algebras was made by Royden ([4]).

The joint spectrum is a basic concept for one of most important parts of the theory of commutative Banach algebras, namely for the operational (symbolic) calculus of analytic functions of several complex variables. Thus the purpose of this paper is to discuss the connection between joint spectra and homomorphisms of commutative Banach algebras in the case when the considered element $x=(x_1, x_2, \cdots, x_n)$ is the n-elements of a commutative Banach algebra with a unit element, e, and then we characterize these relations by the Gel'fand transformations of an analytic functions.

2. Preliminaries

In this section we summarize the general results that will be necessary in the later development, without proofs. So, for more detailed informations, the readers are refered to the indicated references.

Throughout this paper we shall only consider commutative Banach algebras with a unit element, denoted by e. We shall denote by BCe the set of all a commutative Banach algebras with a unit element, e. In what follows we donote the set of regular elements in $A \in BCe$ by $G(A)$, and its complement the set of singular element, by $S(A)$. It is clear that $G(A)$ contains e (unit element), and is a group, and that $S(A)$ contains

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0 (zero element), and we also denote the set of characters, that is, multiplicative linear functionals on $A\in BCe$, that is, a non-zero algebraic homomorphisms of A into the complex field, $\boldsymbol{C}$, by M_A and the kernel of $\phi\in M_A$, i.e., $\ker\phi=\{x\in A:\phi(x)=0\}$ by $\ker\phi$, and the set of maximal ideals in $A\in BCe$ by $M(A)$. The set M_A can naturally be identified with the set $M(A)$ ([2], pp. 78).

Henceforth we denote by $x=(x_1, x_2, \cdots, x_n)$ the n-elements rearrange the lines $x_1, x_2, \cdots, x_n$ of $A\in BCe$ for brevity's sake.

PROPOSITION 1 ([2], [6]). *For* $x=(x_1, x_2, \cdots, x_n)\in A$, *we define the Gel'fand transform* $x^\wedge=(x_1^\wedge, \cdots, x_n^\wedge)$, $x^\wedge: M_A\to\boldsymbol{C}^n$ *by* $x^\wedge(\phi)=\phi(x)$ *for each* $\phi\in M_A$. *Then*

(1) *the mapping* $x\to x^\wedge$ *is a homomorphism of* A *into* $A^\wedge=\{x^\wedge: x\in A\}$

(2) $x^\wedge(\phi)=(x_1^\wedge(\phi), \cdots, x_n^\wedge(\phi))=(\phi(x_1), \cdots, \phi(x_n))$
$=\sigma_A^{\text{joint}}(x)$ *for every* $\phi\in M$.

(3) $x\in S(A)$ *if and only if* $x^\wedge(\phi)=0$ *for some* $\phi\in M_A$.

Rephrasing the above proposition (2), we can say that a point $Z=(\lambda_1, \cdots, \lambda_n)$ belong to $\sigma_A^{\text{joint}}(x_1, \cdots, x_n)$ if and only if $\lambda_1 e-x_1, \cdots, \lambda_n e-x_n$ belong to a common maximal ideal in $A\in BCe$. This occurs if and only if there fail to exist $y_1, \cdots, y_n\in A$ such that $\sum_{i=1}^{n}(\lambda_i e-x_i)y_i=e$.

Following Coburn and Schechter ([3]), we will define the joint spectrum of $x=(x_1, \cdots, x_n)$ in the case of $A\in BCe$ as follows;

$$(2.1)\qquad \begin{aligned}\sigma_A^{\text{joint}}(x)&=\sigma_A^{\text{left}}(x)=\sigma_A^{\text{right}}(x)\\ &=\{(\phi(x_1), \cdots, \phi(x_n))\in\boldsymbol{C}^n:\phi\in M_A\}\end{aligned}$$

This set is the image of the Gel'fand transform ([2], [6]), because $(\phi(x_1), \cdots, \phi(x_n))=(x_1^\wedge(\phi), \cdots, x_n^\wedge(\phi))=x^\wedge(\phi)$ for every $\phi\in M_A$.

PROPOSITION 2 ([5]). *The joint spectrum* $\sigma_A^{\text{joint}}(x)$ *of* $x=(x_1, \cdots, x_n)$ *is a nonempty and a compact subset of* $\boldsymbol{C}^n$, *and the mapping*

$$K_A:\phi\in M_A\to\sigma_A^{\text{joint}}(x)\in\boldsymbol{C}^n$$

is a homeomorphism from M_A *onto* $\sigma_A^{\text{joint}}(x)$.

The following lemma plays a basic role in our development.

LEMMA 3 ([6]). *Let* $x=(x_1, \cdots, x_n)\in A$ *and let* f *be an analytic function in a neighborhood of* $\sigma_A^{\text{joint}}(x)$ *in* $\boldsymbol{C}^n$. *Then*

(2.2) $$f(x)^{\wedge}(\phi)=f(x^{\wedge}(\phi))=f(\phi(x))$$

for every $\phi\in M_A$.

In what follows, we let the symbol $\mathrm{Hol}[\sigma_A{}^{\mathrm{joint}}(x)]$ denote the algebra of all functions analytic on an open set containing the joint spectrum $\sigma_A{}^{\mathrm{joint}}(x)$, and let the symbol $\mathrm{Hom}(A, B)$ denote the set of the unital algebra homomorphisms of $A\in BCe$ into $B\in BCe$, and M_B and M_A denote the set of all multiplicative linear functionals on B and A, respectively. The function $f\in\mathrm{Hol}[\sigma_A{}^{\mathrm{joint}}(x)]$ is said to act on an element $x=(x_1, \cdots, x_n)\in A$ if $f(x^{\wedge}\phi)\in A^{\wedge}$ for every $\phi\in M_A$.

3. Homomorphisms of *BCe*

We now consider that a function $f\in\mathrm{Hol}[\sigma_A{}^{\mathrm{joint}}(x)]$ acts on an element $x=(x_1, \cdots, x_n)$. By (2.1) and (2.2),

(3.1) $$\begin{aligned} f(x)^{\wedge}\phi&=f(x_1{}^{\wedge}(\phi), \cdots, x_n{}^{\wedge}(\phi))=f(\phi(x_1), \cdots, \phi(x_n))\\ &=f(\sigma_A{}^{\mathrm{joint}}(x)) \end{aligned}$$

for every $\phi\in M_A$. From the above result we see that for a $x\in A$ there exists a $y\in A$ such that

(3.2) $$y=f(x)^{\wedge}\phi=f(x^{\wedge}\phi)=f(\phi(x))$$

holds for every $\phi\in M_A$.

Hence we characterize the connection between $f\in\mathrm{Hol}[\sigma_A{}^{\mathrm{joint}}(x)]$ and joint spectra as follows;

THEOREM 4. *Let* $A, B\in BCe$, $h\in\mathrm{Hom}(A, B)$, *and we define a surjective mapping* h^* *of* M_B *onto* M_A *by* $h^*: \varphi\to\varphi\cdot h$ *for every* $\varphi\in M_B$. *If a function* $f\in\mathrm{Hol}[\sigma_A{}^{\mathrm{joint}}(x)]$ *acts on* $x=(x_1, \cdots, x_n)\in A$, *then*

$$f(\sigma_A{}^{\mathrm{joint}}(x))=\sigma_B{}^{\mathrm{joint}}(h(x)) \text{ if and only if } h^*\varphi=\phi f$$

for any $\phi\in M_A$.

Proof. Assumming that $h^*\varphi=\phi f$, from proposition 1(2) and (3.1) we have $(h^*\varphi)(x)=(\varphi h)(x)=\varphi(h(x))=h(x)^{\wedge}\varphi=\sigma_B{}^{\mathrm{joint}}(h(x))$

$$(h^*\varphi)(x)=(\phi f)(x)=\phi(f(x))=f(x)^{\wedge}\phi=f(\sigma_A{}^{\mathrm{joint}}(x))$$

for $x=(x_1, \cdots, x_n)\in A$ and every $\phi\in M_A$, $\varphi\in M_B$.

Thus it follows that $f(\sigma_A{}^{\mathrm{joint}}(x))=\sigma_B{}^{\mathrm{joint}}(h(x))$. Moreover from proposition 1(2) and (3.1) we have

$$f(x)^\wedge\phi=\phi(f(x))=(\phi f)(x)$$
$$=h(x)^\wedge\varphi=\varphi(h(x))=(\varphi h)(x)=(h^*\varphi)(x)$$

for $x=(x_1,\cdots,x_n)\in A$. Hence we conclude that $h^*\varphi=\phi f$ for $x=(x_1,\cdots,x_n)$. This completes the proof.

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Hanyang University
Seoul 133, Korea