J. Korean Math. Soc. 24(1987), No. 2, pp. 267-285

VECTOR BUNDLES ASSOCIATED TO AN AUTOMORPHIC FACTOR

BYUNG MOON JUN AND JAE-HYUN YANG

1. Introduction

Let X be a symmetric bounded domain in C^N and let Γ be a discrete group of holomorphic automorphisms of X with the following properties:

(a) The quotient space $Y=\Gamma\backslash X$ is compact.

(b) Γ acts freely on X.

By Kodaira's theorem (cf. [14]), $Y=\Gamma\backslash X$ is an algebraic manifold. Let X^u be the compact hermtian symmetric manifold which is dual to X in the sense of E. Cartan. We denote by G^c the simply connected covering group of the connected biholomorphic transformations group of X^u and we let U the connected Lie subgroup of G^c fixing a point $x_0^u \in X^u$. Let G^u be the compact real form of G^c. Then $X^u=G^c/U=G^u/K$, where $K=G^u\cap U$. Given a holomorphic representation ρ of the complexification K^c of K into $GL(r;C)$, the composition $J_\rho=\rho\circ J$ of ρ and the canonical automorphic factor J on X (see §2. B, Definition 1) is an automorphic factor of type ρ. This automorphic factor gives rise to a holomorphic vector bundle $E(J_\rho)$ of rank r over $Y=\Gamma\backslash X$. Using a representation ρ of K^c, we get the so-called homogeneous vector bundle $E^u(\rho)$ over X^u in the sense of Bott. In this paper, we prove a vanishing theorem for the cohomology groups $H^q(Y, E(J_\rho))$ under a certain condition on ρ and prove the stability of these vector bundles $E^u(\rho)$ and $E(J_\rho)$.

In section 2, we review the hermitian symmetric manifolds and define the concept of the canonical automorphic factor on a bounded symmetric domain. And we introduce the holomorphic vector bundle $E(J_\rho)$ over $Y=\Gamma\backslash X$ and the homogeneous vector bundle $E^u(\rho)$ over X^u in the sense

Received May 28, 1987.
This work was supported by the Ministry of Education.

of Bott. In section 3 and section 4, we describe the materials which are needed in order to understand section 5. In section 5, we will prove vanishing theorem on $H^q(Y, E(J_\rho))$ under the condition $q_\rho < q$ which is weaker than that of Ise (see Theorem 5.3). In section 6, we prove that the vector bundles $E^u(\rho)$ and $E(J_\rho)$ are stable.

NOTATIONS. i) Lie groups are denoted by the great roman letters G, U, K etc and their Lie algebras by the corresponding small German letters $\mathfrak{g}, \mathfrak{u}, \mathfrak{k}$ etc.
ii) The complexifications of Lie groups (resp. Lie algebras) are denoted by G^c, U^c, K^c etc(resp. $\mathfrak{g}^c, \mathfrak{u}^c, \mathfrak{k}^c$ etc).

2. Preliminaries and hermitian symmetric spaces

A. Let X be a symmetric bounded domain in $\boldsymbol{C}^N$ and X^u the compact symmetric hermitian manifold which is dual to X in the sense of E. Cartan(cf. [4]). The group-theoretical descriptions of X and X^u are as follows: We denote by G^c the simply connected covering group of the connected biholomorphic transformations group of X^u. Then G^c is a connected semisimple complex Lie group and $X^u = G^c/U$, where U is the connected closed Lie subgroup of G^c consisting of all elements of G^c which fix a point x_0^u. A compact form G^u of G^c is also simply connected and acts on X^u transitively as transformations. Therefore X^u can be expressed as $X^u = G^u/K, K = G^u \cap U$. If we set

$$\mathfrak{m} = \{y \in \mathfrak{g}^u : \langle x, y \rangle = 0 \text{ for all } x \in \mathfrak{k}\},$$

then we have

$$\mathfrak{g}^u = \mathfrak{k} + \mathfrak{m} \text{ (direct sum)},$$
$$[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}.$$

If we put

$$\mathfrak{g} = \mathfrak{k} + i\mathfrak{m} \quad (i^2 = -1),$$

then $\mathfrak{g}$ is a noncompact real form of $\mathfrak{g}^c$, the Lie algebra of G^c and $\mathfrak{g}$ generates the real semisimple Lie group G whose center is finite and simple components are all noncompact. G/K is identified with X and $G \cap G^u = K$, $G \cap U = G^u \cap U = K$. Therefore if we define the mapping j of X into X^u by

$$j : gK \longrightarrow gU \quad (g \in U),$$

then j becomes an injection of X into X^u which is compatible with the actions of G on X and X^u. Thus X will be endowed with the G-invariant complex structure from that of the open submanifold $j(X)$ of X^u.

Let x_0 be the point of X corresponding to K. We may identify $i\mathfrak{m}$ with the tangent vector space T_{x_0} of X at the point x_0. The complex structure I on X defines a linear transformation I_x in each tangent vector space T_x $(x\in X)$ such that $I_x^2=-1$. Let T_x^c be the complexification of T_x. Each $u\in T_x^c$ is written uniquely in the form $u=v+iw$ where $v, w\in T_x$. Put $\bar{u}=v-iw$. We have $T_x^c=T_x^+ + T_x^-$, $\overline{T_x^+}=T_x^-$, $T_x^+\cap T_x^-=(0)$, where T_x^+ (resp. T_x^-) denotes the i (resp. $(-i)$)-eigenspace of I_x. And we have $\mathfrak{g}\mathfrak{k}^c=\mathfrak{k}^c+\mathfrak{m}^c$, $\mathfrak{k}^c\cap\mathfrak{m}^c=(0)$ and idenified with $T_{x_0}^c$. Hence $\mathfrak{m}^c$ decomposes into direct sum

$$\mathfrak{m}^c=\mathfrak{n}^+ + \mathfrak{n}^-, \quad \mathfrak{n}^-=\overline{\mathfrak{n}^+}.$$

We see that each complex vector field $Y\in\mathfrak{n}^+$ (resp. $Y\in\mathfrak{n}^-$) is characterized by the property that $\pi_0 Y_s\in T^+_{x_0(s)}$ (resp. $\pi_0 Y_s\in T^-_{x_0(s)}$) for every $s\in G$, where π_0 is the projection of G onto $X=G/K$. In fact, I defines a linear isomorphism I_0 of $i\mathfrak{m}$ which commutes with the adjoint action of K on $i\mathfrak{m}$ and $I_0^2=-1$. $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) is the i-eigenspace (resp. $(-i)$-eigenspace) of I_0. It is known that

$$(1.1)\qquad \begin{aligned} &[\mathfrak{n}^+, \mathfrak{n}^+]=(0), \quad [\mathfrak{n}^-, \mathfrak{n}^-]=(0), \\ &[\mathfrak{k}^c, \mathfrak{n}^+]\subset\mathfrak{n}^+, \quad [\mathfrak{k}^c, \mathfrak{n}^-]\subset\mathfrak{n}^-. \end{aligned}$$

Now we know that $\mathfrak{k}$ contains a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let Δ be the root system of $\mathfrak{g}^c$ with respect to $\mathfrak{h}^c$, and $\mathfrak{g}_\alpha$ the root space of $\mathfrak{g}^c$ corresponding to the root $\alpha\in\Delta$. Let $u\longrightarrow\bar{u}$ be the conjugation of $\mathfrak{g}^c$ with respect to the real form $\mathfrak{g}$. Then we have $\overline{\mathfrak{g}_\alpha}=\mathfrak{g}_{-\alpha}$ and $\dim_C\mathfrak{g}_\alpha=1$ for all $\alpha\in\Delta$. By (1.1), we can easily see that

$$\mathfrak{n}^+=\textstyle\sum_{\alpha\in\phi}\mathfrak{g}_\alpha, \quad \mathfrak{n}^-=\sum_{\alpha\in\phi}\mathfrak{g}_\alpha,$$

where ϕ is a subset of Δ. A root α is called a complementary root if $\mathfrak{g}_\alpha\subset\mathfrak{n}^+ + \mathfrak{n}^-$. We may choose an ordering for the roots so that the roots belonging to ϕ are all positive. We fix such an ordering for the roots once and for all. A root belonging to ϕ is called a positive complementary root. Let (,) be the Cartan-Killing form of $\mathfrak{g}^c$. For each $\alpha\in\phi\cup(-\phi)$, choose an element $X\in\mathfrak{g}$ such that $(X_\alpha, X_{-\alpha})=1$. We have then $\overline{X}_\alpha=X_{-\alpha}$. Moreover it is clear that either $\mathfrak{g}_\alpha\subset\mathfrak{k}^c$ or $\mathfrak{g}_\alpha\subset\mathfrak{m}^c$. In the first case the root is called compact, and in the second case noncompact. And we have the decompositions

$$\mathfrak{k}^c=\mathfrak{h}^c+\sum_\alpha \mathfrak{g}_\alpha \ (\alpha : \text{compact}), \quad \mathfrak{m}^c=\sum_\beta \mathfrak{g}_\beta \ (\beta : \text{noncompact}).$$

Let $\mathfrak{u}$ be the Lie algebra of U. Let N^+ (resp. N^-) be the connected Lie subgroup of G^c corresponding to $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) and K^c the complexification of K. We know $\mathfrak{n}^-\subset\mathfrak{u}$ and $U=K^cN^-$ (the semidirect product).

B. We shall define the notion of the canonical automorphic factor. We first review the results of Harish-Chandra (cf. [5] or [6]). G^c contains $N^+K^cN^-$ as an open subset, and the mapping of the complex manifold $N^+\times K^c\times N^-$ into G^c defined by $(n^+, k, n^-)\longrightarrow n^+kn^- (n^+\in N^+$, $k\in K$, $n^-\in N^-)$ is a biholomorphic mapping of $N^+\times K^c\times N^-$ onto the open submanifold $N^+K^cN^-$ of G^c and G is contained in $N^+K^cN^-$. Hence we see that GU is an open subset of N^+U and N^+U is that of G^c. Using $G\cap U=K$ and $N^+\cap U=\{1\}$, we then have

$$X=G/K \subset N^+ \subset G^c/U=X^*.$$

We shall denote by j_1 (resp. j_2) the first (resp. the second) inclusion mapping; they are holomorphic. Since G is contained in $N^+K^cN^-$, each $g\in G$ is written uniquely in the form $g=n^+kn^-$ with $n^+\in N^+$, $k\in K^c$, $n^-\in N^-$. Put $n^+=n^+(g)$. Then $n^+(gk)=n^+(g)$ for all $k\in K$ and the mapping N of G/K into N^+ defined by

$$N(x)=n^+(g), \quad x=gK=gx_0\in X,$$

is a holomorphic and bijective mapping of X onto an open bounded subset of the complex manifold N^+ with a suitable metric. We see easily that $g^{-1}N(gx_0)\in U$ for all $g\in G$.

LEMMA 1. *$N(gx)^{-1}(gN(x))\in U$ for all $g\in G$ and $x\in X$.*

Proof. Let $x=g'x_0$, $g'\in G$. Then $(gg')^{-1}N(gg'x_0)=(gg')^{-1}N(gx)\in U$. $g^{-1}N(x)=g^{-1}N(gx_0)\in U$. Therefore $N(gx)^{-1}(gN(x))=((gg')^{-1}N(gx))^{-1}g^{-1}N(x)\in U$. This proves the lemma.

By Lemma 1, $gN(x)$ is written uniquely in the form

$$gN(x)=N(gx)\ J(g,x)n',$$

where $J(g,x)\in K^c$ and $n'\in N^-$. Then J is a C^∞-mapping of $G\times X$ into K^c and $J(g,\cdot)$ is holomorphic in the variable $x\in X$. The mapping $J: G\times X\longrightarrow K^c$ satisfies the following properties:

(i) $J(gg',x)=J(g,g'x)J(g',x)$ for all $g,g'\in G$ and $x\in X$.

(ii) $J(g,x_0)$ isthe K^c-component of $g\in N^+K^cN^-$ and $J(k,x)=k$ for every $k\in K^c$ and $x\in X$.

(iii) $J(1, x)=1$ for all $x \in X$, and $J(g, x)^{-1}=J(g^{-1}, gx)$ for every $g \in G$ and $x \in X$.

DEFINITION 1. The mapping $J: G \times X \longrightarrow K^c$ introduced above is called the *canonical automorphic factor* on the symmetric bounded domain $X=G/K$.

DEFINITION 2. A C^∞ mapping R of $G \times X$ into the complex general linear group $GL(r : \boldsymbol{C})$ $(r \geqq 1)$ is called an *automorphic factor* if

(i) $R(g, x)$ is holomorphic in the variable $x \in X$.

(ii) $R(gg', x)=R(g, g'x)\ R(g', x)$ for all $g, g' \in G$ and $x \in X$.

A $\boldsymbol{C}^r$-valued holomorphic function $f(x)$ on X is called an *automorphic form* with respect to the automorphic factor R if $f(gx)=R(g, x)\ f(x)$ for all $g \in G$ and $x \in X$.

Given a representation ρ of K^c in the complex vector space $\boldsymbol{C}^r$, we now define the mapping $J_\rho=\rho \circ J : G \times X \longrightarrow GL(r\,;\boldsymbol{C})$ by

$$J_\rho(g, x)=\rho(J(g, x)), \quad g \in G, \quad x \in X.$$

Then J_ρ is an automorphic factor in the sense of the above definition and is called the *automorphic factor* of *type* ρ.

Let Γ be a discrete subgraup of G which satisfies the following two conditions:

(a) The quotient space $\Gamma \backslash X$ is compact.

(b) Every element $\gamma \in \Gamma$ different from the identity 1 has no fixed point in X.

By Kodaira's theorem (cf. [14]), $Y=\Gamma \backslash G/K$ is an algebraic manifold. A holomorphic mapping f of X into $\boldsymbol{C}^r$ is called an automorphic form of type ρ with respect to Γ if $f(\gamma, x)=J_\rho(\gamma, x) f(x)$ for all $\gamma \in \Gamma$ and $x \in X$. Now we introduce the holomorphic vector bundle $E(J_\rho)$ over $Y=\Gamma \backslash X$ as follows: We define the action of G on the trivial vector bundle $X \times \boldsymbol{C}^r$ over X by

$$g \circ (x, u)=(gx, \quad J_\rho(g, x) u)$$

for $g \in G$, $x \in X$ and $u \in \boldsymbol{C}^r$, and then the quotient manifold $(X \times \boldsymbol{C}^r)/\Gamma$ is a holomorphic vector bundle over $Y=\Gamma \backslash X$. This holomorphic vector bundle will be denoted by $E(J_\rho)$ and is called the vector bundle over Y defined by the automorphic factor J_ρ. We can see easily that an automorphic form of type ρ can be identified with a holomorphic cross section of $E(J_\rho)$ and conversely.

C. We recall the concept of homogeneous vector bundle over X^u (cf. [3]). Every holomorphic representation ρ of K^c in $\boldsymbol{C}^r$ can be extended naturally to the holomorphic representation of $U=K^cN^-$ on $\boldsymbol{C}^r$, which we also will denote by ρ. Conversely for every completely reducible holomorphic representation $\rho: U \longrightarrow GL(r; \boldsymbol{C})$, we can show that $\rho(N^-)=1$. So we may consider it as a representation of K^c. From now on we denote by $\mathrm{Hom}(K^c, GL(r; \boldsymbol{C}))$ (resp. $\mathrm{Hom}\,(U, GL(r; \boldsymbol{C}))$) the set of all holomorphic representations of K^c (resp. U) on $\boldsymbol{C}^r$. For any $\rho \in \mathrm{Hom}(K^c, GL(r; \boldsymbol{C}))$, we define $E^u(\rho)$ as the quotient manifold $(G^c \times \boldsymbol{C}^m)/U$ of $G^c \times \boldsymbol{C}^m$ by U under the actions such that

$$s\circ(g, u) = (gs^{-1}, \ \rho(s)u)$$

for all $g \in G^c$, $u \in \boldsymbol{C}^r$ and $s \in U$. Then $E^u(\rho)$ has a holomorphic vector bundle structure over X^u with the fibre $\boldsymbol{C}^r$. We recall $E^u(\rho)$ the *homogeneous vector bundle* over X^u with respect to ρ.

LEMMA 2. *The vector bundle $E(J_\rho)$ is holomorphically equivalent to $\Gamma\backslash j^*E^u(\rho)$, where $j: X \longrightarrow X^u$ is the mapping defined by $j(gK)=gU$ for every $g \in G$.*

EXAMPLE. Let Θ_Y (resp. K_Y) be the tangent bundle (resp. the canonical line bundle) over Y. We denote by Θ^u and K^u the tangent bundle and the canonical line bundle over X^u. Then $\Gamma\backslash j^*\Theta^u=\Theta_Y$ and $\Gamma\backslash j^*K^u =K_Y$. R. Bott teaches us that $\Theta^u=E^u(Ad_K)$ and $K^u=E^u(\delta_K{}^{-1})$, where $Ad_K: K \longrightarrow GL(\mathfrak{n}^+)$ is the adjoint representation of K on $\mathfrak{n}^+$ and $\delta_K \in \mathrm{Hom}\,(K^c, \boldsymbol{C}^*)$ is the character whose differential is the sum of all positive complementary roots. By Lemma 2, $\Theta_Y=E(Ad_K\circ J)$ and $K_Y =E(\delta_K{}^{-1}\circ J)$.

D. Let $(\ ,\ )$ be the Cartan-Killing form of $\mathfrak{g}^c$. Since the restriction of $(\ ,\ )$ to $\mathfrak{h}^c \times \mathfrak{h}^c$ is nondegenrate, there is a natural isomorphism of $(\mathfrak{h}^c)^*=\mathrm{Hom}\,(\mathfrak{h}^c, \boldsymbol{C})$ onto $\mathfrak{h}^c$. For any $\lambda \in (\mathfrak{h}^c)^*$ its image will be denoted by H_λ. Thus

$$\lambda(H)=(H, H_\lambda), \ \ \lambda \in (\mathfrak{h}^c)^*, \ \ H \in \mathfrak{h}^c$$

can define a symmetric nondegenerate bilinear form $\langle\ ,\ \rangle$ on $(\mathfrak{h}^c)^* \times (\mathfrak{h}^c)^*$ by $(\lambda, \ \tau)=(H_\lambda, H_\tau)$, $\lambda, \tau \in (\mathfrak{h}^c)^*$. We note that $H_\alpha \neq 0$ for any root $\alpha \in \Delta$ and $[X_\alpha, \ X_{-\alpha}]=[X_\alpha, \ X_{\bar{\alpha}}]=H_\alpha$ for $\alpha \in \Delta$. We set, for any root

$\alpha\in\Delta$,

$$\bar{H}_\alpha=\frac{2H_\alpha}{(\alpha,\alpha)}.$$

Then $[X_\alpha, X_{\bar\alpha}]=(\alpha,\alpha)/2\cdot\bar{H}_\alpha$ for any root $\alpha\in\Delta$. For a simple root $\alpha_i(1\leqq i\leqq 1)$, we set $\bar{H}_{\alpha_i}=\bar{H}_i$ $(1=\dim_C \mathfrak{h}^c)$. A weight λ on $\mathfrak{h}^c$ is a linear form on $\mathfrak{h}^c$ such that $\lambda(\bar{H}_i)\,(1\leqq i\leqq 1)$ are all integers. The weight Λ_i $(1\leqq i\leqq 1)$ such that $\Lambda_i(\bar{H}_j)=\delta_{ij}$ are called the *fundamental dominant weights.* We put $\delta=\sum_{\alpha>0}\alpha$. Then $\delta=2\sum_{i=1}^{l}\Lambda_i$. We denote by δ_K the sum of all complementary positive roots. Let Ξ be the set of all simple roots $\alpha\in\Delta$ such that $X_\alpha\in[\mathfrak{h}^c,\mathfrak{h}^c]$. We put $\delta_M=2\sum_{\alpha_i\in\Sigma}\Lambda_i$.

3. The complexes $A(\Gamma, X, \rho)$ and $A(\Gamma, X, J_\tau)$

Let X be a symmetric bounded domain in $\boldsymbol{C}^N$. As in the previous section, we may write $X=G/K$ where G is a connected semisimple Lie group and K a maximal compact subgroup of G. Let Γ be a discrete subgroup of G acting on X freely such that the quotient space $Y=\Gamma\backslash X$ is compact. Let ρ be a representation of G in a complex vector space F. We denote by $A^p(\Gamma, X, \rho)$ the vector space of all F-valued smooth p-forms η on X such that

$$\eta\circ L_\gamma=\rho(\gamma)\eta$$

for all $\gamma\in\Gamma$, where L_γ is the translation of X by γ. The graded module $A(\Gamma, X, \rho)=\sum_p A^p(\Gamma, X, \rho)$ is a complex with the coboundary operator defined by the exterior differentiation d. We denote by $H^p(\Gamma, X, \rho)$ the cohomology groups of the complex $A(\Gamma, X, \rho)$. Let $\mathfrak{w}: G\longrightarrow\Gamma\backslash G$ be a projection. Then $\mathfrak{w}$ defines an injection of $\mathfrak{g}$ into the Lie algebra of all vector fields on $\Gamma\backslash G$ because Γ is a discrete subgroup of G. From now on we shall identify the Lie algebra $\mathfrak{g}$ with its image by this injection so that $A\in\mathfrak{g}$ will be identified with the vector field $\mathfrak{w}(A)$ on $\Gamma\backslash G$.

Let η be a form in $A^p(\Gamma, X, \rho)$. If $\pi: G\longrightarrow X$ is a projection of G onto X, we define a form η° on G by putting $\eta_s^\circ=\rho(s^{-1})(\eta\circ\pi)_s$ for each $s\in G$. The form η° is invariant under Γ and therefore η° may be considered as a form on $\Gamma\backslash G$. The image $A_0^p(\Gamma, X, \rho)$ of $A^p(\Gamma, X, \rho)$ by the mapping $\eta\rightarrow\eta^\circ$ consists of all F-valued p-forms on $\Gamma\backslash G$ satisfying the following conditions

$$(3.1)\qquad \begin{cases}\theta(Z)\eta^\circ+\rho(Z)\eta^\circ=0,\\ i(Z)\eta^\circ=0\end{cases}$$

for all $Z\in\mathfrak{k}$, where $\theta(Z)$ and $i(Z)$ denote the operators of Lie derivation and interior product by the vector field Z respectively. The graded module $A(\Gamma, X, \rho)$ is bigraded; $A(\Gamma, X, \rho) = \sum_r A^r(\Gamma, X, \rho) = \sum_{p,q} A^{p,q}(\Gamma, X, \rho)$. Here $A^{p,q}(\Gamma, X, \rho)$ consists of the (p, q)-forms on X. We know that $H^r(\Gamma, X, \rho)$ decomposes into the direct sum $\sum_{p+q} H^{p,q}(\Gamma, X, \rho)$, where $H^{p,q}(\Gamma, X, \rho)$ is the subgroup of $H^r(\Gamma, X, \rho)$ consisting of the elements represented by closed (p, q)-forms ([16]). Let $A_0^{p,q}(\Gamma, X, \rho)$ be the submodule of $A_0^{p+q}(\Gamma, X, \rho)$ corresponding to $A^{p,q}(\Gamma, X, \rho)$ by the isomorphism $\eta\to\eta^\circ$. A form $\eta^\circ\in A_0^{p+q}(\Gamma, X, \rho)$ belongs to $A_0^{p,q}(\Gamma, X, \rho)$ if and only if the following condition is satisfied; if $\eta^\circ(X_1, \ldots, X_{p+q})\neq 0$ with $X_i\in\mathfrak{n}^{\pm}$, then the number of X_i belonging to $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) equals p (resp. q) (see[16], p. 400).

For a holomorphic representation τ of K^c in a complex vector space S, i. e., $\tau\in\mathrm{Hom}\,(K^c, GL(S))$, we define the canonical automorphic factor of type τ by

$$J_\tau(g, x) = \tau(J(g, x)),\quad g\in G,\quad x\in X,$$

where J_τ is the canonical automorphic factor on $X=G/K$ (see § 2. B). Let $A^r(\Gamma, X, J_\tau)$ (resp. $A^{p,q}(\Gamma, X, J_\tau)$) be the vector space of all S-valued r-forms (resp. (p, q)-forms) on X such that

$$(\eta\circ L_\gamma)_x = J_\tau(\gamma, x)\eta_x$$

for all $x\in X$ and $\gamma\in\Gamma$. We have $A^r(\Gamma, X, J_\tau) = \sum_{p+q=r} A^{p,q}(\Gamma, X, J_\tau)$. We set $A(\Gamma, X, J_\tau) = \sum_{p,q} A^{p,q}(\Gamma, X, J_\tau)$. Then the operator d'' defines a coboundary operator of type $(0, 1)$ in $A(\Gamma, X, J_\tau)$. We now denote by $H_{d''}^{p,q}(\Gamma, X, J_\tau)$ the cohomology groups of this complex $(A(\Gamma, X, J_\tau),\ d'')$.

For a form $\eta\in A^{p,q}(\Gamma, X, J_\tau)$, we define a $(p+q)$-form η° on G by setting $\eta^\circ = J_\tau(s, x_0)^{-1}(\eta\circ\pi)_s$, where $s\in G$, $x_0=\pi(e)$, e the identity of G. Then η° is induced by the projection $\mathfrak{w} : G\to\Gamma\backslash G$ from an S-valued $(p+q)$-form on $\Gamma\backslash G$, which we also denote by η°. The mapping $\eta\to\eta^\circ$ maps the module $A^{p,q}(\Gamma, X, J_\tau)$ bijectively onto the module $A_0^{p,q}(\Gamma, X, J_\tau)$ consisting of all S-valued (p, q)-forms on $\Gamma\backslash G$ such that

$$(3.2)\qquad \begin{cases} \theta(Z)\eta^\circ + \tau(Z)\eta^\circ = 0, \\ i(Z)\eta^\circ = 0 \end{cases}$$

for all $Z\in\mathfrak{k}^c$, and that if $\eta^\circ(X_1, \ldots, X_{p+q})\neq 0$ with $X_i\in\mathfrak{n}^{\pm}$, then the number of X_i belonging to $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) equals p (resp. q).

Proposition 3.1 ([16], p. 408). *Every cohomolgy class of* $H_{d''}^{p,q}(\Gamma, X,$

J_τ) is represented by a unique harmonic (p, q)-form on X.

4. The complex of a Lie algebra

Let $\mathfrak{g}^c$ be a complex Lie algebra and $\mathfrak{k}^c$ a subalgebra of $\mathfrak{g}^c$. If F is a $\mathfrak{g}^c$-module, the action of $\mathfrak{g}^c$ on F is denoted by $X\cdot f\,(X\in\mathfrak{g}^c,\ f\in F)$. Let $C(\mathfrak{g}^c;F)=\sum C^c(\mathfrak{g}^c;F)$, $n=0, 1, 2, \ldots$, be the standard complex of $(\mathfrak{g}^c, F)$. That is, $C^0(\mathfrak{g}^c;F)=F$ and $C^p(\mathfrak{g}^c;F)$ is the vector space of all p-linear alternating forms on $\mathfrak{g}^c$ with values in F. On $C(\mathfrak{g}^c;F)$, we can define the operators $\theta(X)$ and $i(X)$ of Lie derivation and interior product by $X\in\mathfrak{g}^c$ as follows:

$$\theta(Z)f=Z\cdot f,$$

$$(4.1)\qquad (\theta(Z)\eta)(X_1,\ldots,X_p)=Z\cdot\eta(X_1,\ldots,X_p) -\textstyle\sum_{u=1}^{p}\eta(X_1,\ldots,[Z,X_u],\ldots,X_p),$$

where $Z\in\mathfrak{g}^c$, $f\in C^\circ(\mathfrak{g}^c;F)$, $\eta\in C^p(\mathfrak{g}^c;F)$ $(p\geqq 1)$, while

$$i(Z)\eta=0 \text{ for } \eta\in C^0(\mathfrak{g}^c;F)=F,$$

$$(i(Z)\eta)(X_1,\ldots,X_{p-1})=\eta(Z,X_1,\ldots,X_{p-1})$$

for $Z\in\mathfrak{g}^c$, $\eta\in C^p(\mathfrak{g}^c;F)\,(p\geqq 1)$. Then there exists a unique operator d of degree 1 such that $i(Z)d+di(Z)=\theta(Z)$ for all $Z\in\mathfrak{g}^c$ and d is given by

$$(d\eta)(X_1,\ldots,X_{p+1})=\textstyle\sum_{u=1}^{p+1}(-1)^{u+1}X_u\eta(X_1,\ldots,\hat{X}_u,\ldots,X_{p+1}) +\sum_{u<v}(-1)^{u+v}\eta([X_u,X_v],X_1,\ldots,\hat{X}_u,\ldots,\hat{X}_v,\ldots,X_{p+1}).$$

Since $d^2=0$, the module $C(\mathfrak{g}^c;F)$ is a complex with coboundary operator d. We call this complex the cochain complex of $\mathfrak{g}^c$ with coefficients in the $\mathfrak{g}^c$-module F. We denote by $H^p(\mathfrak{g}^c;F)$ the cohomology groups of this complex. Let $C^p(\mathfrak{g}^c,\mathfrak{k}^c;F)$ be the subspace of $C^p(\mathfrak{g}^c;F)$ consisting of all elements $\eta\in C^p(\mathfrak{g}^c;F)$ such that $\theta(X)\eta=i(X)\eta=0$ for all $X\in\mathfrak{k}^c$. The submodule $C(\mathfrak{g}^c,\mathfrak{k}^c;F)=\sum_p C^p(\mathfrak{g}^c,\mathfrak{k}^c;F)$ is stable under d and thus a subcomplex of $C(\mathfrak{g}^c;F)$. We shall this complex $C(\mathfrak{g}^c,\mathfrak{k}^c;F)$ the *cochain complex* of $\mathfrak{g}^c$ *relative to* $\mathfrak{k}^c$ *with coefficients* in F and the cohomology groups of this complex will be denoted by $H^p(\mathfrak{g}^c,\mathfrak{k}^c;F)$

From now on we assume that $X=G/K$ is a symmetric bounded domain in $\boldsymbol{C}^N$. We will resume the notation in §2. Let $\mathcal{D}$ be the vector space of all complex valued C^∞-functions on $\Gamma\backslash G$ and let $\mathcal{F}=\mathcal{D}\otimes_C F$, where F is a finite dimensional complex vector space. Then the $\mathfrak{g}^c$-module structure is given on $\mathcal{F}$ by defining

$$m(Z)(f\otimes u)=Zf\otimes u + f\otimes\rho(Z)u$$

for all $Z\in\mathfrak{g}^c$, $f\in\mathcal{D}$ and $u\in F$. Here ρ is a representation of G^c in a complex vector space F. Let $C(\mathfrak{g}^c ; F)=\sum_p C^p(\mathfrak{g}^c ; \mathcal{F})$, and $\theta(Z)$ (resp. $i(Z)$) be the derivation (resp. the interior product) by Z in $C(\mathfrak{g}^c ; \mathcal{F})$. Let $C(\mathfrak{g}^c, \mathfrak{k}^c ; \mathcal{F})$ be the cochain complex of $\mathfrak{g}^c$ relative to $\mathfrak{k}^c$ with coefficients in $\mathcal{F}$. We can easily see that the complex $A_0(\Gamma, X, \rho)$ may be identified with the complex $C(\mathfrak{g}^c, \mathfrak{k}^c ; \mathcal{F})$. Therefore the complex $A(\Gamma, X, \rho)$ will be identified with the complex $C(\mathfrak{g}^c, \mathfrak{k}^c ; \mathcal{F})$ and so the cohomology groups $H^p(\Gamma, X, \rho)$ with the relative cohomology groups $H^p(\mathfrak{g}^c, \mathfrak{k}^c ; \mathcal{F})$. Since $\mathfrak{m}^c=\mathfrak{n}^+\oplus\mathfrak{n}^-$ (cf. §2. B), the complex $C(\mathfrak{g}^c, \mathfrak{k}^c ; \mathcal{F})$ is then bigraded; $C^{p,q}(\mathfrak{g}^c, \mathfrak{k}^c ; \mathcal{F})$ consists of all $C^r(\mathfrak{g}^c, \mathfrak{k}^c ; \mathcal{F})$ $(r=p+q)$ such that if $(Y_1, \ldots, Y_r)\neq 0$ and $Y_i\in\mathfrak{n}^\pm$ for $i=1,2,\ldots,r$, then the number of Y_i belonging to $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) equals p (resp. q). We easily see that $C^{p,q}(\mathfrak{g}^c, \mathfrak{k}^c ; \mathcal{F})$ may be identified with $A^{p,q}(\Gamma, X, \rho)$. Hence $H^{p,q}(\Gamma, X, \rho)$ can be identified with $H^{p,q}(\mathfrak{g}^c, \mathfrak{k}^c ; \mathcal{F})$.

We now identify the module $C^{p,q}(\mathfrak{g}^c, \mathfrak{k}^c ; \mathcal{F})$ with a subspace of $\mathcal{F}\overset{p}{\otimes}\mathfrak{n}^+\overset{q}{\otimes}\mathfrak{n}^-$ in the following way. Let $\Psi=\{\alpha_1, \ldots, \alpha_N\}$ be the set of the positive complementary roots and let $\{X_\alpha ; \alpha\in\Psi\}$ and $\{X_{\bar\alpha} ; \alpha\in\Psi\}$ be the eigenvectors of roots as introduced in §2. We put $X_i=X_{\alpha_i}$ and $X_{\bar i}=X_{\bar\alpha_i}$ for $\alpha_i\in\Psi$. Since $\{X_i\}$ and $\{X_{\bar i}\}$ are bases of $\mathfrak{n}^+$ and $\mathfrak{n}^-$ respectively and $(X_i, X_j)=\delta_{ij}$, $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) may be identified with the dual of $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$). We define $l : C^{p,q}(\mathfrak{g}^c, \mathfrak{k}^c ; \mathcal{F}) \longrightarrow \mathcal{F}\overset{p}{\otimes}\mathfrak{n}^+\overset{q}{\otimes}\mathfrak{n}^-$ by setting

$$l(\eta)=\sum_{i_1<\cdots<i_p}\sum_{\bar j_1<\cdots<\bar j_q}\eta(X_{i_1}, \ldots, X_{i_p}, X_{\bar j_1}, \ldots, X_{\bar j_q}) \otimes\textstyle\bigwedge_{t=1}^{p} X_{i_t}\otimes\bigwedge_{s=1}^{q} X_{\bar j_s}.$$

We simply write $\eta(X_{i_1}, \ldots, X_{i_p}, X_{\bar j_1}, \ldots, X_{\bar j_q})=\eta_{i_1\cdots i_p\bar j_1\cdots\bar j_q}$.

If we restrict the representation ρ of $\mathfrak{g}^c$ onto the abelian subalgebra $\mathfrak{n}^-$, we may regard F as an $\mathfrak{n}^-$-module. Let $C(\mathfrak{n}^-, F)=\sum_q C^q(\mathfrak{n}^-, F)$ be the cochain complex of $\mathfrak{n}^-$ with coefficients in F, and denote by d^- its coboundary operator. We identify $C(\mathfrak{n}^-, F)=\sum_q C^q(\mathfrak{n}^-, F)$ with $F\otimes\bigwedge\mathfrak{n}^+ =\sum_q F\otimes\bigwedge^q\mathfrak{n}^+$. Then we have

$$d^-=\sum_{k=1}^{N}\rho(X_{\bar k})\otimes\varepsilon(X_k),$$

where $\varepsilon(X)$ denotes the exterior multiplication by X. We define a positive definite hermitian inner product in $C(\mathfrak{n}^-, F)$ by

$$(c, c')=\sum_q\sum_{i_1<\cdots<i_q}(c_{\bar i_1\cdots\bar i_q}, c'_{\bar i_1\cdots\bar i_q})_F,$$

where $c=\sum_q\sum_{i_1<\cdots<i_q}c_{\bar i_1\cdots\bar i_q}\otimes(X_{i_1}\wedge\ldots\wedge X_{i_q})$ and $c'=\sum_q\sum_{i_1<\cdots<i_q}$

$c'_{\bar{i}_1 \dots \bar{i}_q} \otimes (X_{i_1} \wedge \dots \wedge X_{i_q})$ and $(\, , \,)_F$ is an admissible inner product on F (cf. [16], p. 375). Let δ^- be the adjoint operator of d^- with respect to this inner product and we define a Laplacian Δ^- by

$$\Delta^- = d^- \delta^- + \delta^- d^-.$$

We call a cocycle c of $C(\mathfrak{n}^-, F)$ *harmonic* if $\Delta^- c = 0$. Every cohomology class in $H^q(\mathfrak{n}^-, F)$ is represented by a unique harmonic cocycle (cf. [17]).

5. Vanishing theorems for the cohomology groups $H^q(Y, E(J_\tau))$

First we recall Bott's results concerning the induced representations with respect to homogeneous vector bundles (cf. [3]). Suppose that $\rho \in \mathrm{Hom}\ (K^c, GL(r\,;\, \boldsymbol{C}))$ is an irreducible representation. The action of $g \in G^c$ on $E^u(\rho)$ as bundle isomorphism induces the linear automorphism on the $\boldsymbol{C}$-module $H^q(X^u, E^u(\rho))$, which we will write $\rho^{(q)}(g)$. The representation $(\rho^{(q)}, H^q(X^u, E^u(\rho))$ of G^c thus obtained is by definition the q-th induced representation of ρ. Let Λ be the highest weight of ρ. Let Λ be the highest weight of ρ. Then the induced representations $\rho^{(q)} (0 \leqq q \leqq N)$ are determined only by Λ:

THEOREM 5.1 (Bott, [3]). *If there exists a root* $\alpha \in \Delta$ *such that* $(\Lambda + \frac{1}{2}\delta, \alpha) = 0$, *then all* $\rho^{(q)}$ $(0 \leqq q \leqq N)$ *are the the* 0*-representations. Otherwise, there is one and only one induced representation* $\rho^{(q)}$ *which is irreducible and its highest weight* Λ' *is given by*

$$\Lambda' + \frac{1}{2}\delta = \varepsilon(\Lambda + \frac{1}{2}\delta),$$

where ε *is the element of the Weyl group which is the product of* q *reflections with respect to the simple root planes* $\alpha_i = 0$ *and it is uniquely determined by the condition* $(\Lambda' + \frac{1}{2}\delta, \alpha_i) > 0 (1 \leqq i \leqq l)$.

We define λ and μ in $\mathrm{Hom}(\mathfrak{h}^c, \boldsymbol{C})$ as follows (cf. §2. D).

$$\Lambda = \lambda + \mu, \quad \lambda = \sum_{\alpha_i \notin \Xi} m_i \Lambda_i, \quad \mu = \sum_{\alpha_j \in \Xi} m_j \Lambda_j.$$

Ise (cf. [10]) obtained the following results using Bott's results ([3]).

THEOREM 5.2. i) *Suppose that*

$$(\Lambda - \mu - \delta_M, \alpha) > 0$$

for all complementary positive roots α. *Then*

$$H^q(Y, E(J_\rho))=0 \quad \textit{for all } q<N.$$

ii) *Suppose that*

$$(\Lambda+\delta, \alpha)<0$$

for all complementary positive roots α. *Then*

$$H^q(Y, E(J_\rho))=0 \quad \textit{for all } q>0.$$

THEOREM 5.3. *If there exists at least one complementary positive root* α *such that* $(\Lambda, \alpha)>0$, *then*

$$H^0(Y, E(J_\rho))=0.$$

Matsushima and Murakami(cf. [17]) showed the following:

THEOREM 5.4. *Let* q_ρ *be the number of roots* $\alpha\in\Psi$ *such that* $(\Lambda, \alpha)>0$. *Then*

$$H^{0,q}_{d''}(\Gamma, X, J_\rho)=(0) \quad \textit{for } q_\rho<q.$$

If $(\Lambda, \alpha_i)>0$, *for* $i=1, \ldots, s$, *where* $\alpha_1, \ldots, \alpha_s$ are the simple roots belonging to Ψ, then

$$H^{p,q}_{d''}(\Gamma, X, J_\rho)=(0) \quad \textit{for } q<N.$$

Here Ψ *denotes the set of all positive complimentary roots.*

THEOREM 5.5. *Suppose that* $(\Lambda, \alpha)>0$ *for all positive roots* α *of* $\mathfrak{g}^c$. *Then for all* $p+q=N$,

$$H^{p,q}(\Gamma, X, \rho)=(0).$$

THEOREM 5.6. *Let* $\tilde{\Lambda}$ *be the lowest weight of* ρ. *Let* p_ρ *be the number of roots* α *in* Ψ *such that* $(\tilde{\Lambda}, \alpha)<0$. *Then the cohomology group* $H^{p,0}(\Gamma, X, \rho)$ *vanishes for* $p<p_\rho$.

REMARK 1. By a theorem of Hirzebruch [8], we have

$$\chi(Y)=p_a(Y)\chi(X^u),$$

where $p_a(Y)$ denotes the arithmetic genus of Y and $\chi(Y)$ (resp. $\chi(X_u)$) is the Euler characteristic of the complex manifold Y (resp. the compact form X^u of X). Moreover, Hirzebruch [7] gave the following formula

$$\chi(Y)=(-\pi)^{-N}d_N\, v(Y),$$

where X is irreducible, $v(Y)$ denotes the total volume of Y with respect to the Bergman metric on X and

$$d_N=\frac{\prod_{\alpha\in\Psi}(\frac{1}{2}\delta,\alpha)}{(2(\delta_K,\gamma))^N}\chi(X^u).$$

Here γ denotes the unique simple root belonging to Ψ. It is known that $2(\delta_K,\gamma)=1$. Thus we get

$$d_N=\prod_{\alpha\in\Psi}\left(\frac{1}{2}\delta,\alpha\right)\chi(X^u).$$

Hence we obtain

$$p_a(Y)=(-\pi)^{-N}\prod_{\alpha\in\Psi}\left(\frac{1}{2}\delta,\alpha\right)v(Y).$$

By Weyl's formula,

$$r=\prod_{\alpha>0}\frac{\left(\Lambda+\frac{1}{2}\delta,\alpha\right)}{\left(\frac{1}{2}\delta,\alpha\right)}.$$

By the way, we have(cf. [19])

$$\sum_{p=0}^{2N}(-1)^p\dim_C H^p(\Gamma,X,\rho)=r\chi(Y).$$

Theorem 5.5 yields the following

$$H^p(\Gamma,X,\rho)=(0)\ \text{ if } p\neq N.$$

$$\dim_C H^N(\Gamma,X,\rho)=(-\pi)^{-N}\frac{\prod_{\alpha>0}\left(\Lambda+\frac{1}{2}\delta,\alpha\right)}{\prod_{\alpha\in\Xi}\left(\frac{1}{2}\delta,\alpha\right)}\chi(X^u)v(Y).$$

REMARK 2. If ρ is irreducible, we let $\rho=\rho_1\oplus\cdots\oplus\rho_k$ denote the decomposition of ρ into irreducible components. We have $E(J_\rho)=E(J_{\rho_1})\oplus\ldots\oplus E(J_{\rho_k})$ ($\oplus$ denotes the Whitney sum). Hence we have

$$H^q(Y,E(J_\rho))=\sum_{i=1}^{k}H^q(Y,E(J_{\rho_i})),\quad 0\leq q\leq N.$$

REMARK 3. It is known (cf. [10]) that

$$\chi(Y,E(J_\rho))=\chi(Y)\chi(X^u,E^u(\rho)).$$

If X i sirreducible, $\chi(Y)=(-\pi)^{-N}d_N v(Y)$ (cf. Remark 1). $\chi(X^u,E^u(\rho))$ can be computed by Theorem 5.1. Finally we can compute $\chi(Y,E(J_\rho))$

THEOREM A. *Let q_ρ the number of roots $\alpha\in\Psi$ such that $(\Lambda,\alpha)>0$. Then*

$$H^q(Y,E(J_\rho))=(0)\ \textit{for}\ q_\rho<q.$$

Moreover, if $(\Lambda, \alpha)>0$ *for every simple root* α *belonging to* Ψ, *then*

$$H^{p,q}(Y, E(J_\rho))=(0) \text{ for } q>N.$$

Here $N=\dim_C X$ *and* Ψ *is the set of all positive complementary roots.*

Proof. Let $\pi : X \to \Gamma\backslash X=Y$ be the projection of X onto Y and let $\varphi : X\times C^r \longrightarrow (X\times C^r)/\Gamma=E(J_\rho)$ be the canonical projection of $X\times C^r$ onto $E(J_\rho)$. For each $x\in X$, let φ_x be the linear isomorphism of the typical fibre C^r onto the fibre $E(J_\rho)_z$ of $E(J_\rho)$ over the point $z=\pi(x)\in Y$ defined by

$$\varphi_x(x)=\varphi(x, \xi), \quad \xi\in C^r.$$

For each $\alpha\in A^{p,q}(\Gamma, X, J_\rho)$, we define the (p,q)-form θ on Y with coefficients in $E(J_\rho)$ as follows:

$$\theta_z(\pi Z_1, \dots, \pi Z_p, \pi W_1, \dots, \pi W_q)=\varphi_x\alpha_x(Z_1, \dots, Z_p, W_1, \dots, W_q),$$

where $x\in X$, $z=\pi(x)$, $Z_1, \dots, Z_p\in T_x^+(X)$ and $W_1, \dots, W_q\in T_x^-(X)$. The mapping $\alpha\to\theta$ yields an isomorphism of the bigraded module $A(\Gamma, X, J_\rho)$ onto the bigraded module $A(Y, (J_\rho))=A(E(J_\rho))=\sum_{p,q}A^{p,q}(E(J_\rho))$. Thus the cohomology $H_{d''}^{p,q}$ (Γ, X, J_ρ) is isomorphic to the cohomology $H_{d''}^{p,q}$ $(Y, E(J_\rho))$. But the following exact sequence

$$0^0 \longrightarrow \Omega^p(E(J_\rho)) \longrightarrow A^{p,0}(E(J_\rho)) \xrightarrow{d''} A^{p,1}(E(J_\rho)) \xrightarrow{d''} \cdots$$

is a fine resolution of the sheaf $\Omega^p(E(J_\rho))$ of thegerms of holomorphic p-forms with coefficients in $E(J_\rho)$. Thus we have (cf. [9], p36)

$$H^q(Y, \Omega^p(E(J_\rho)))\cong H_{d''}^{p,q}(Y, E(J_\rho)) \quad \text{(Dolbeault)}.$$

Hence by Theorem 5.4, we have for $p=0$

$$H^q(Y, E(J_\rho))=H_{d''}^{0,q}(Y, E(J_\rho))=H_{d''}^{0,q}(\Gamma, X, J_\rho)=(0)$$

for $q_\rho<q$. The second assertion follows immediately from Theorem 5.4 and the above argument. Q. E. D.

REMARK4. Let $\mathfrak{g}^c$ be a simple Lie algebra over C and let γ be the unique simple root of $\mathfrak{g}^c$ belonging to Ψ. Let ad be the adjoint representation of $\mathfrak{g}^c$ and $\rho=ad_+$ the representation of K^c in $\mathfrak{n}^-$. Then $H_{d''}^{p,q}$ (Γ, X, J_ρ) is isomorphic to $H^q(Y,\Theta)$, where Θ is the tangent bundle of Y. It is known that $q_\rho<\frac{1}{(\gamma,\gamma)}-1$. Hence the cohomology group H^q

(Y,Θ) vanishes for $q<\frac{1}{(\gamma,\gamma)}-1$ (cf. [4]).

6. Stability and Einstein condition of $E(J_\rho)$ and $E^u(\rho)$

Before we study the stability of the vector bundle $E(J_\rho)$ and $E^u(\rho)$, we first review the concept of stability and Hermitian-Einstein structure on vector bundles.

Let E be a holomorphic vector bundle of rank r over a compact Kähler manifold X. For a Hermitian metric h along the fibre of E, the Hermitian connection, $D: A^0(E) \longrightarrow A^1(E)$ is characterized by the properties

(a) $d(h(s,t))=h(Ds,t)+h(s,Dt)$, $s,t\in A^0(E)$,

(b) $D''s=d''s$, where D'' denotes the $(0,1)$-component of D.

With respect to a local frame $\{e_\alpha\}$, the connection matrix $A=(A_\alpha^\beta)$ $(1\leqq\alpha,\beta\leqq n)$ is given by

$$A_\alpha^\beta=(d'h_{\alpha\bar\gamma})h^{\gamma\bar\beta},$$

where $h_{\alpha\bar\beta}=h(e_\alpha,e_\beta)$ and $(h^{\alpha\bar\beta})=(h_{\alpha\bar\beta})^{-1}$.

The curvature $F=dA-A\wedge A$ of the Hermitian connection for a holomorphic vector bundle reduces to the $(1,1)$-form with coefficients in $\mathrm{End}(E)$

$$F=d''A=h^{-1}d''d'h+h^{-1}d'h\wedge h^{-1}d''h.$$

Conversely, the integrability theorem of Newlander-Nirenberg implies that a complex vector bundle admits a holomorphic structure if there exists a $U(r)$ connection whose curvature is of type $(1,1)$.

Given a Kähler metric g on X, we define an operation tr_g: $A^{1,1}(\mathrm{End}(E))\longrightarrow A^0(\mathrm{End}(E))$ as follows. For a section $F=(F_\alpha^\beta)\in A^{1,1}(\mathrm{End}(E))$,

$$tr_gF=(\sum g^{j\bar k}F_{\alpha j\bar k}^\beta)_{1\leq\alpha,\beta\leq n}=\sum_{j,k}g^{j\bar k}F_{j\bar k},$$

where $F_\alpha^\beta=F_{\alpha j\bar k}^\beta dz^j\wedge d\bar z^k$ and $F_{j\bar k}=(F_{\alpha j\bar k}^\beta)_{1\leq\alpha,\beta\leq r}$.

Definition 1. A holomorphic vector bundle of rank r over a compact Kähler manifold (X,g) is said to be *Hermitian-Einstein* if there exists a Hermitric h for which the Hermitian curvature F satisfies:

$$tr_gF=\mu I,$$

where I is the identity endomorphism of E and μ is a constant.

Let $\mathcal{F}$ be a torsion-free coherent sheaf over a compact Kähler manifold (X, g) of dimension n. Let ω be the Kähler form; it is a real positive closed $(1,1)$-form on X. Let $c_1(\mathcal{F})$ be the first Chern class of $\mathcal{F}$, that is, the first Chern class of the determinant bundle $\det(\mathcal{F})$ over X. It is represented by a real closed $(1,1)$-form on X. The degree of $\mathcal{F}$ is defined to be

$$\deg(\mathcal{F}) = \int_M c_1(\mathcal{F}) \wedge \omega^{n-1}.$$

The degree/rank ratio or slope $\mu(\mathcal{F})$ is defined to be

$$\mu(\mathcal{F}) = \deg(\mathcal{F})/\mathrm{rank}(\mathcal{F}).$$

DEFINITION 2. A coherent sheaf $\mathcal{F}$ over a compact Kähler manifold (X, g) is said to be *stable* (resp. *semi-stable*) if for every coherent sheaf $\mathcal{F}'$ of lower rank, $\mu(\mathcal{F}') < \mu(\mathcal{F})$ (resp. $\leqq$).

REMARKS. (i) $\mathcal{F}$ is reflexive if and only if $\mathcal{F}^{**} = (\mathcal{F}^*)^* = \mathcal{F}$. A reflexive sheaf of rank one is a holomorphic line bundle.
(ii) A reflexive sheaf is locally free outside a subvariety of codimension greater than or equal to 2.
(iii) The dual $\mathcal{F}^*$ of any coherent sheaf $\mathcal{F}$ is reflexive.
(iv) $\mathcal{F}$ is (semi-) stable if and only if its dual $\mathcal{F}^*$ is (semi-) stable.

Kobayashi (cf. [12]) obtained the following differential geometical criterion for stability.

THEOREM (Kobayashi). *Let E be a holomorphic vector bundle over a compact Kähler manifold (X, g) with a Kähler form ω. If E admits an irreducible Hermitian-Einstein connection, then E is stable.*

The converse of the above theorem was known as Kobayashi's conjecture. Donaldson proved Kobayashi's conjecture in the case X is an algebraic surface. Quite recently Uhlenbeck and Yau (cf. [24]) proved Kobayashi's conjecture in the case X is of higher dimension.

THEOREM (Uhlenbeck and Yau). *A stable holomorphic vector bundle over a compact Kähler manifold admits a unique Hermitian-Einstein connection.*

We are now in a position to prove the stability of the vector bundles

$E^u(\rho)$ and $E(J_\rho)$. Ramanan ([22]), Umemura ([23]), and Kobayashi ([13]) indepently showed the following:

THEOREM (Ramanan, Umemura and Kobayashi). *Let G^c be a simply connected, semisimple complex Lie group and U a parabolic subgroup without simple factor. Let ρ be a finite dimensional irreducible holomorphic representation of U. Then the homogeneous vector bundle over $M=G^c/U$ defined by a representation ρ is H-stable for any ample line bundle H.*

In § 2. C, we mentioned the homogeneous vector bundle $E^u(\rho)$ over X^u, the compact hermitian symmetric manifold which is dual to a bounded symmetric domain $X=G/K$ in the sense of E. Cartan. X^u can be expressed as $X^u=G^c/U$ (see § 2. A). It is well-known that X^u is an algebraic manifold. Thus U is a parabolic subgroup of the simply connected semisimple complex Lie group G^c. Hence by the above theorem, $E^u(\rho)$ is stable. By Uhlenbeck and Yau, $E^u(\rho)$ admits a unique Hermitian-Einstein connection.

Let $(\ ,\ )$ be the standard inner product in $\boldsymbol{C}^r$. Since K^c is the complexification of a compact Lie group K, there exists a hermitian inner product $\langle \ ,\ \rangle$ in $\boldsymbol{C}^r$ which is invariant under $\rho(K^c)$. This defines canonically a hermitian metric in the fibres of $E(J_\rho)$ as follows. On the fibre $E(J_\rho)_z$ over $z\in Y$, we define

$$\langle [x,\xi],[x,\eta]\rangle_z=\langle \xi,\eta\rangle,\quad x\in X,\quad z=\pi(x),\quad \xi,\eta\in \boldsymbol{C}^r,$$

where $\pi: X\longrightarrow Y=\Gamma\backslash X$ is a projection and $[x,\xi]$ is the equivalence class of (x,ξ) in $X\times\boldsymbol{C}^r$, i.e., $[x,\xi]\in E(J_\rho)_z$. It is well defined. Indeed, for each $\gamma\in\Gamma$, $x\in X$, ξ, $\eta\in\boldsymbol{C}^r$,

$$\begin{aligned}
&\langle[\gamma x, J_\rho(\gamma,x)\xi],\ [\gamma x, J_\rho(\gamma,x)\eta]\rangle_z\\
&=\langle J_\rho(\gamma,x)\xi,\ J_\rho(\gamma,x)\eta\rangle\\
&=\langle\rho(J(\gamma,x))\xi,\rho(J(\gamma,x))\eta\rangle\\
&=\langle\xi,\eta\rangle\ \ (\text{since } J(\gamma,x)\in K^c)\\
&=\langle[x,\xi],[x,\eta]\rangle_z.
\end{aligned}$$

Thus this hermitian metric gives rise to a flat structure on $E(J_\rho)$. Hence it admits an irreducible Hermitian-Einstein connetion and so $E(J_\rho)$ is stable.

Summarizing what we have proved, we state

THEOREM B. *Let ρ be an irreducible holomorphic representation of K^c into $GL(r;\boldsymbol{C})$. Then $E^u(\rho)$ is H-stable for any ample line bundle H*

over X^u *and* $E(J_\rho)$ *admits an irreducible flat Hermitian-Einstein connection.*

REMARK. Let X be a bounded symmetric domain in $\boldsymbol{C}^N$ and let Γ be a neat arithmetic group on X. Then $Y=\Gamma\backslash X$ is a smooth quasi-projective algebraic variety. Consider the case $Y=\Gamma\backslash X$ is not compact. Given a representation ρ of K, we have then a holomorphic vector bundle $E(J_\rho)$ over Y. We obtain a smooth projective compactification $\bar{Y}$ by the toroidal compactification and thus get the corresponding vector bundle $\bar{E}(J_\rho)$ over $\bar{Y}$. We refer to D. Mumford [20] for detalis. The following problem is still open.

PROBLEM. Is $\bar{E}(J_\rho)$ stable? In other words, does $\bar{E}(J_\rho)$ admit a Hermitian-Einstein connection?

Bibliography

1. A. Borel, *On the curvature tensor of the hermitian symmetric manifolds,* Ann. of Math. **71**(1960), 508-521.
2. A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces,* Part I, Amer. J. Math. **80**(1958), 459-538.
3. R. Bott, *Homogeneous vector bundles,* Ann. of Math. vol. **66**(1957), 203-248.
4. Calabi and E. Vesentini, *On compact, locally symmetric Kähler manifolds,* Ann. of Math. **71**(1960), 427-507.
5. Harish-Chandra, *Representations of semisimple Lie groups on a Banach space VI,* Amer. J. Math., **78**(1956), 564-628.
6. Helgason, *Differential Geometry and Symmetric Spaces,* Academic Press, New York, 1978.
7. F. Hirzebruch, *Characteristic numbers of homogeneous domains,* Seminars on Analytic Functions, vol. **2**, Princeton, 1957, 92-104.
8. F. Hirzebruch, *Automorphe Formen und der Satz von Riemann-Roch,* Symposium internacional de Topologia Algebraica, Mexico, 1958, 129-144.
9. F. Hirzebruch, *Topological Methods in Algebraic Geometry,* Springer-Verlag, 1978.
10. M. Ise, *Generalized automorphic forms and certain holomorphic vector bundles,* Amer. J. Math. **86**(1964), 70-108.
11. S. Kobayashi and K. Nomizu, *Foundation of Differential Geometry,* John-Wiley & Sons, New York, 1969.
12. S. Kobayashi, *Curvature and stability of vector bundles,* Proc. Japan Acad. **58**(1982), 158-162.

13. S. Kobayashi, *Homogeneous vector bundles and stability*, Nagoya Math. J. **101** (1986), 37-54.
14. K. Kodaira, *On Kähler varieties of restricted type*, Ann. of Math. **60** (1954), 28-46.
15. B. Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math. **74** (1961), 329-387.
16. Y. Matsushima and S. Murakami, *On vector bundle valued harmonic forms and automorphic forms on symmetric riemannian manifolds*, Ann. of Math. **78** (1963), 365-416.
17. Y. Matsushima and S. Murakami, *On certain cohomology groups attached to hermitian symmetric spaces*, Osaka J. Math. **2** (1965), 1-35.
18. Y. Matsushima and S. Murakami, *On certain cohomology groups attached to hermitian symmetric spaces* (II), Osaka J. Math. **5** (1968), 223-241.
19. Y. Matsushima and G. Shimura, *On the cohomology groups attached to certain vector valued differential formson the product of the upper half planes*, Ann. of Math. **78** (1963), 417-449.
20. D. Mumford, *Hirzebruch's proportionality theorem in the noncompact case*, Inven. Math. **42** (1977), 239-272.
21. C. Okonek, M. Schneider and H. Spindler, *Vector bundles over complex projective spaces*, Progress in Math. **3** (1980), Birkhäuser.
22 S. Ramanan, *Holomorphic vector bundles on homogeneous spaces*, Topology **5** (1966), 159-177.
23 H. Umemura, *On a theorem of Ramanan*, Nagoya Math. J. **69** (1978), 131-138.
24 K. Uhlenbeck and S.T. Yau, *On the existence of Hermitian Yang-Mills connections in stable vector bundles*, Comm. on Pure and Applied Math., vol. **XXXIX** S257-S293 (1986).

Inha University
Incheon 160, Korea