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SOME BANACH ALGEBRAS OF YEH-FEYNMAN INTEGRABLE FUNCTIONALS*

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1. Introduction

Let $f(s,t)$ be a real valued function on $Q=[a,b]\times[c,d]$ and let $R=[a', b']\times[c', d']$ be a subrectangle of Q and $\Delta_R(f)=f(b',d')-f(a',d')-f(b',c')+f(a',c')$. A function $f(s,t)$ is absolutely continuous on Q ($f\in AC(Q)$) if the following two conditions are satisfied; (i) given $\varepsilon>0$, there exists $\delta>0$ such that $\sum_{R\in\mathcal{J}}|\Delta_R(f)|<\varepsilon$ whenever $\mathcal{J}$ is a finite collection of pairwise non-overlapping subrectangles of Q with $\sum_{R\in\mathcal{J}} m(R)<\delta$, where m denotes Lebesgue measure on $\boldsymbol{R}^2$,
(ii) the function $f(.,d)$ and $f(b,.)$ are absolutely continuous functions of a single variable on $[a,b]$ and $[c,d]$, respectively.

Let $C_2\equiv C_2(Q)$ be the Yeh-Wiener space on $Q=[a,b]\times[c,d]$, that is, the space of real valued continuous functions $x(s,t)$ on Q such that $x(s,c)=x(a,t)=0$ for all $(s,t)\in Q$. Let $D_2\equiv D_2(Q)$ be the class of elements $x\in C_2(Q)$ such that $x\in AC(Q)$ and $\dfrac{\partial^2 x(s,t)}{\partial s\,\partial t}\in L_2(Q)$, where $L_2\equiv L_2(Q)$ is a real Hilbert space of Lebesgue measurable, real valued, square integrable functionals on Q.

Let $\mathcal{A}$ be the σ-algebra of subsets of $L_2(Q)$ generated by the class of sets of the form

$$\{v\in L_2 : \int_Q v(s,t)\phi(s,t)\,dsdt<\lambda\}$$

where $\phi\in L_2$ and $\lambda\in\boldsymbol{R}$. The above σ-algebra $\mathcal{A}$ is actually the Borel class of L_2, that is, the σ-algebra $\mathcal{B}(L_2)$ generated by the norm open subsets of L_2 [9]. Let $M\equiv M(L_2(Q))$ be the class of complex measures of finite variation defined on $\mathcal{B}(L_2)$. If $\mu\in M$, we set $\|\mu\|=\text{var}\ \mu$ over

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L_2. Let $BV(Q)$ be the class of all real valued functions which are of bounded variation on Q [6]. A property that holds except on a scale-invariant null set in $C_2(Q)$ is said to hold scale-invariant almost everywhere (s-a. e.). A function F is said to be scale-invariant measurable provided F is defined on a scale-invariant measurable set and $F(\rho x)$ is Yeh-Wiener measurable for every $\rho>0$. Two functionals F and G on $C_2(Q)$ are said to be equal s-a. e. $(F\approx G)$ if for each $\rho>0$, the equation $F(\rho x)=G(\rho x)$ holds for a. e. x in $C_2(Q)$. For a discussion of scale-invariant measurability in $C_2(Q)$ see [5].

The definitions of the Banach algebras $S, \hat{S}, S^*, S$ with which we will be concerned throughout involves the Paley-Wiener-Zygmund (P. W. Z.) integral, a simple type of stochastic integral, for functions of two real variables which we now define.

Let $\{\phi_j\}$ be a complete orthonormal (C. O. N.) set of real valued functions of bounded variation on Q. Let $v\in L_2(Q)$ and

$$v_n(s, t)=\sum_{j=1}^{n}\left(\int_Q v(p, q)\phi_j(p, q)\,dp\,dq\right)\phi_j(s, t).$$

Then the P. W. Z. integral is defined by the formula

$$\int_Q v(s,t)\widetilde{dx}(s,t)=\lim_{n\to\infty}\int_Q v_n(s, t)\,dx(s, t)$$

for all $x\in C_2(Q)$ for which the limit exists, where the integral $\int_Q v_n(s, t)dx(s, t)$ means the Riemann-Stieltjes integral. For a nice discussion of the n-dimensional Riemann-Stieltjes integrals see [13].

Now we introduce the binary quadratic approximation. Let m be a non-negative integer and consider the division of $Q=[a,b]\times[c,d]$ into subrectangles by means of the partition $\sigma(m)$;

$$a=s_0<s_1<\dots<s_{2^m}=b,\ c=t_0<t_1<\dots<t_{2^m}=d,$$

where $s_j=a+\frac{j(b-a)}{2^m}$, $t_k=c+\frac{k(d-c)}{2^m}$ for $j,\ k=0, 1, \dots, 2^m$.

For each $x\in C_2(Q)$, we define the m^{th} binary quadratic approximation $x_{\sigma(m)}$ by formula

$$\begin{aligned}x_{\sigma(m)}(s,t)=&\frac{x(s_j,t_k)-x(s_{j-1},t_k)-x(s_j,t_{k-1})+x(s_{j-1},t_{k-1})}{(s_j-s_j)(t_k-t_{k-1})}(s-s_{j-1})\\&(t-t_{k-1})+\frac{x(s_j,t_{k-1})-x(s_{j-1},t_{k-1})}{s_j-s_{j-1}}(s-s_{j-1})+\\&\frac{x(s_{j-1},t_k)-x(s_{j-1},t_{k-1})}{t_k-t_{k-1}}(t-t_{k-1})+x(s_{j-1},t_{k-1})\end{aligned}$$

for $(s, t) \in [s_{j-1}, s_j] \times [t_{k-1}, t_k]$ for $j, k = 1, 2, \ldots, 2^m$.

Using the binary quadratic approximation, we now make a definition which parallels Cameron and Storvick's definition [3] of continuity with respect to binary polygonal approximation.

Definition. A functional F on $C_2(Q)$ is said to be continuous at x with respect to binary quadratic approximation if $\lim_{m\to\infty} F(x_{\sigma(m)}) = F(x)$, where $x_{\sigma(m)}$ is the m^{th} binary quadratic approximation of x, and "$\lim_{m\to\infty}$" means that m approaches ∞ so that $\|\sigma(m)\| \to 0$.

We note that if F is continuous on $C_2(Q)$ then it is certainly continuous with respect to binary quadratic approximation at every x in $C_2(Q)$

The purpose of this paper is to present three Banach algebras $\hat{S}, S^*$, and S' of functionals on Yeh-Wiener space which are similar to those on Wiener space that Cameron and Storvick have treated in [2] and [3]. Furthermore, we examine the above Banach algebras on Yeh-Wiener space, and prove how they are related to the Banach algebra S in [6]. Finally we show that S^* is intermediate between S' and $S \cap \hat{S}$, and it is closely related to $\hat{S}$.

2. Preliminaries and Some simple results

In this section, we give definitions for the spaces of Yeh-Wiener functionals $S, \hat{S}, S^*, S'$, and present some propositions which will be used in the final section.

Definition 2.1. Let $S \equiv S(L_2)$ be the space of functionals F expressible in the form

$$(2.1) \qquad F(x) = \int_{L_2} \exp\ \{i \int_Q v(s,t) \tilde{d}x(s,t)\}\, d\mu(v)$$

for s-a. e. x in C_2, where $\mu \in M$.

The following proposition is a well known result. We will state it without proof [4].

Proposition 2.1. *If $F \in S$, there is a unique measure $\mu \in M$ such that*

$$F(x) = \int_{L_2} \exp\{i \int_Q v(s,t) \tilde{d}x(s,t)\}\, d\mu(v)$$

for a.e. x in C_2. Further this equation provides a one-to-one correspondence between M and S. Finally if $F, G \in S$ and $F(x)=G(x)$ for a.e. x, then $F(x)=G(x)$ for s-a.e. x in C_2.

DEFINITION 2.2. The functional F defined on a subset of $C_2(Q)$ that contains $D_2(Q)$ is said to be an element of $\hat{S} \equiv \hat{S}(L_2)$ if there exists a measure $\mu \in M$ such that for $x \in D_2(Q)$,

$$(2.2) \qquad F(x)=\int_{L_2} \exp\{i\int_Q v(s,t)\frac{\partial^2 x(s,t)}{\partial s\,\partial t}dsdt\}\,d\mu(v).$$

NOTATION. If $F(x)=G(x)$ for s-a.e. x in $C_2(Q)$ and for every x in $D_2(Q)$, we shall write $F \cong G$.

From Theorem 4 of [11], we have that if $v \in L_2(Q)$ and $x \in D_2(Q)$ then

$$(2.3) \qquad \int_Q v(s,t)\tilde{d}x(s,t)=\int_Q v(s,t)\frac{\partial^2 x(s,t)}{\partial s\,\partial t}dsdt.$$

Thus if $v \in L_2(Q)$ and $\{\phi_n\}$, $\{\psi_n\}$ are two C.O.N. sequences of $BV(Q)$, then

$$^{\{\phi_n\}}\int_Q v(s,t)\tilde{d}x(s,t)={}^{\{\psi_n\}}\int_Q v(s,t)\tilde{d}x(s,t)$$

for $x \in D_2(Q)$ and hence

$$\int_{L_2} \exp\{i\,^{\{\phi_n\}}\int_Q v(s,t)\tilde{d}x(s,t)\}\,d\mu(v) \cong \int_{L_2} \exp\{i\,^{\{\psi_n\}}\int_Q v(s,t)\tilde{d}x(s,t)\}\,d\mu(v).$$

We now introduce the class of functionals S^*.

DEFINITION 2.3. Let $S^* \equiv S^*(L_2)$ be the space of functionals F expressible in the form

$$(2.4) \qquad F(x)=\int_{L_2} \exp\{i\int_Q v(s,t)\tilde{d}x(s,t)\,d\mu(v)$$

for s-a.e. $x \in C_2(Q)$ and for every $x \in D_2(Q)$, where $\mu \in M$.

Let $BV' \equiv BV'(Q)$ be the class of real valued, right-upper continuous functions (in the sense that $\lim_{s\to s', t\to t'} v(s,t)=v(s',t')$ for $s>s', t>t'$) of bounded variation on Q that vanish at $(.,d)$ and $(b,.)$. Note that the Borel class $\mathcal{B}(BV')$ of BV' is just $\mathcal{B}(L_2) \cap BV'$. Let $M' \equiv M'(BV')$ be the class of complex measures of finite variation defined on $\mathcal{B}(BV')$.

If $\mu \in M'$, we set $\|\mu\| = \text{var } \mu$ over BV'.

Definition 2.4. Let $S' \equiv S'(BV')$ be the space of functionals of the form

$$F(x) = \int_{BV'} \exp \{ i \int_Q v(s,t) dx(s,t) \} d\mu(v) \tag{2.5}$$

for $x \in C_2(Q)$, where $\mu \in M'$

The following proposition is a well known result. We will state it without proof [10].

Proposition 2.2. *Let $v \in L_2(Q)$. If*

$$F(x) = \int_Q v(s, t) \tilde{d}x(s, t) \tag{2.6}$$

then for s–a. e. $x \in C_2(Q)$ and everywhere $x \in D_2(Q)$, $F(x)$ is continuous with respect to binary quadratic approximation.

Proposition 2.3. *If $F \in S^*$, then $F(x)$ is continuous with respect to binary quadratic approximation for s–a. e. $x \in C_2(Q)$ and everywhere $x \in D_2(Q)$.*

Proof. Since $F \in S^*$, there exists $\mu \in M$ such that (2.4) holds. Then substituting $x_{\sigma(m)}$ for x, we have

$$F(x_{\sigma(m)}) = \int_{L_2} \exp \{ i \int_Q v(s,t) \widetilde{dx_{\sigma(m)}}(s,t) \} d\mu(v).$$

By Proposition 2.2, the above exponential approaches the exponential in (2.4) as $m \to \infty$, so by the bounded convergence theorem and because the exponential is measurable in (v, x) on $L_2 \times C_2$, we have $F(x_{\sigma(m)}) \to F(x)$ for s–a. e. $x \in C_2(Q)$ and all $x \in D_2(Q)$.

Corollary 2.1. *If $F, G \in S^*$ and $F(x) = G(x)$ for all binary quadratic functions in $C_2(Q)$, then $F \cong G$.*

Corollary 2.2. *If $F \in \hat{S}$ and F is defined only on $D_2(Q)$, then there exists an extension $F^* \in S^*$ such that $F^*(x) = F(x)$ on $D_2(Q)$. Moreover F^* is essentially unique in the sense that if $F^*, F^{**} \in S^*$ and $F^*(x) = F^{**}(x) = F(x)$ on $D_2(Q)$, then $F^* \cong F^{**}$. Finally if μ is associated with F by (2.2), it follows that μ is associated with F^* by (2.4).*

Remark 1. Let $v \in BV(Q)$ and let $x \in D_2(Q)$. Then the following Riemann–Stieltjes integral and Lebesgue integral are equal [8].

$$\int_Q v(s,t)\,dx(s,t)=\int_Q v(s,t)\frac{\partial^2 x(s,t)}{\partial s\,\partial t}dsdt.$$

3. The Spaces of Functionals $\hat{S}, S^*$, and S'

The purpose of this section is to show that the spaces of functionals $\hat{S}, S^*$, and S' are Banach algegras with the proper norm and to establish their relationships. The proofs of Propositions 3.2, 3.6 and 3.8 are identical with those of Theorems 3.2 and 3.3 of [4]. We will skip those proofs.

PROPOSITION 3.1. *If $F\in S^*$ and F is given by* (2.4) *with $\mu\in M$, it follows that μ is uniquely determined by F.*

Proof. Since $F\in S^*\subset S$, $\mu\in M$ is uniquely determined by (2.1). But since $F\in S^*$, (2.4) holds for $\mu\in M$ with the stronger relation $\cong$ and thus μ is uniquely determined.

We have known that S is a Banach algebra with the norm $\|F\|=\|\mu\|$ [4]. And so for $F\in S^*$, we define $\|F\|=\|\mu\|$.

PROPOSITION 3.2, *The space S^* is a Banach algebra.*

PROPOSITION 3.3. *$S'\subset S$.*

Proof. Let $F\in S'$. Then there exists $\mu'\in M'$ such that (2.5) holds. Since $v\in BV'$, $v\in BV$, so $BV'\subset L_2$. Let $E\in\mathcal{B}(L_2)$ and $E'=E\cap BV'$. Then $E'\in\mathcal{B}(BV')$. Let us define a measure on $L_2(Q)$ by

(3.1) $\qquad \mu(E)=\mu'(E\cap BV')$ for all $E\in\mathcal{B}(L_2)$.

Now
$$\begin{aligned}F(x)&=\int_{BV'}\exp\left\{i\int_Q v(s,t)\,dx(s,t)\right\}d\mu'(v)\\&=\int_{BV'}\exp\left\{i\int_Q v(s,t)\,\tilde{d}x(s,t)\right\}d\mu'(v)\\&=\int_{L_2}\exp\left\{i\int_Q v(s,t)\,\tilde{d}x(s,t)\right\}d\mu(v)\end{aligned}$$
for s-a.e. $x\in C_2(Q)$.

PROPOSITION 3.4. *If $F\in S'$ and F is given by* (2.5) *with $\mu'\in M'$, it follows that μ' is uniquely determined by F.*

Proof. This follows from the Proposition 2.1.

PROPOSITION 3.5. *If $F\in S'$ and μ' is a measure in M' related to F*

by (2. 5), *then* $\|F\|=\|\mu'\|$.

Proof. By definition, $\|F\|=\|\mu\|$ where μ is the unique element of M related to F by (2. 1). Then it follows that $\mu(E)=\mu'(E\cap BV')$ for $E\in \mathscr{B}(L_2)$. Now

$$\operatorname*{var}_{L_2} \mu=\operatorname*{var}_{BV'} \mu',$$

so $\|\mu\|=\|\mu'\|$.

Proposition 3. 6. *The space* S' *is a Banach algebra.*

Proposition 3. 7. *If* $F\in\hat{S}$ *and* F *is given by* (2. 2) *with* $\mu\in M$, *it follows that* μ *is uniquely determined by* F *on* D_2.

Proof. Let F_1 be the restriction of F to D_2. Then $F_1\in\hat{S}$. By Corollary 2. 2, there exists an essentially unique $F^*\in S^*$ such that $F_1(x)=F^*(x)$ on D_2. Since the measure defining F^* is unique, the measure $\mu\in M$ satisfying (2. 2) is unique.

Remark 2. If $F\in\hat{S}$, we define $\|F\|=\|\mu\|$, where μ is associated with F by (2. 2) for $x\in D_2(Q)$. If follows from Proposition 3. 7 that for $F\in\hat{S}$ the measure μ is uniquely determined by F and it is clear that $\|F\|$ is a norm for $\hat{S}$ if we identify elements of $\hat{S}$ which are equal on $D_2(Q)$.

Proposition 3. 8. *The space* $\hat{S}$ *is a Banach algebra, where elements of* $\hat{S}$ *that are equal on* D_2 *are considered equivalent.*

Now we present our main theorem.

Theorem. $S'\subsetneqq S^*\subsetneqq S\cap\hat{S}$.

Proof. For $F\in S'$, there exists $\mu'\in M'$ such that

$$F(x)=\int_{BV'} \exp\{i\int_Q v(s,t)dx(s,t)\}d\mu'(v)$$

for $x\in C_2(Q)$. Just as in the proof of Proposition 3. 3, we define a measure μ on $L_2(Q)$ as follows: Let $\mu(E)=\mu'(E\cap BV')$ for $E\in\mathscr{B}(L_2)$. Let $x\in D_2(Q)$. Then by (2. 3) and Remark 1, we have for $v\in BV(Q)$,

$$\int_Q v(s,t)\tilde{d}x(s,t)=\int_Q v(s,t)\frac{\partial^2 x(s,t)}{\partial s\,\partial t}dsdt=\int_Q v(s,t)dx(s,t).$$

Then for $x\in D_2(Q)$,

$$\begin{aligned}F(x)&=\int_{BV'}\exp\{i\int_Q v(s,t)dx(s,t)\}d\mu'(v)\\&=\int_{BV'}\exp\{i\int_Q v(s,t)\frac{\partial^2 x(s,t)}{\partial s\,\partial t}dsdt\}d\mu'(v)\\&=\int_{L_2}\exp\{i\int_Q v(s,t)\frac{\partial^2 x(s,t)}{\partial s\,\partial t}dsdt\}d\mu(v)\\&=\int_{L_2}\exp\{i\int_Q v(s,t)\tilde{d}x(s,t)\}d\mu(v).\end{aligned}$$

By Proposition 3.3, the first and last members above are equal for s-a.e. $x\in C_2(Q)$, so that $S'\subset S^*$ and $S^*\subset S\cap\hat{S}$ by definition.

We now present an example which shows $S^*\neq S\cap\hat{S}$ Let

$$F(x)=\begin{cases}0 & \text{if } x\in D_2\\ 1 & \text{if } x\in C_2-D_2.\end{cases}\tag{3.2}$$

Then since D_2 is a scale-invariant null set, we have $F\approx 1$, and hence $F\in S$. Clearly F is also an element of $\hat{S}$. By Proposition 2.3, if F were an element of S^*, it would be continuous with respect to binary quadratic approximation s-a.e. on $C_2(Q)$, and it would be therefore zero s-a.e. on $C_2(Q)$, since it is zero for all elements of $D_2(Q)$. This contradicts the fact that it is unity s-a.e. and so $F\notin S^*$ and hence $S^*\neq S\cap\hat{S}$.

Now finally, we shall show that $S'\neq S^*$. First of all, given μ' in M', we define $I\mu'=\mu$ as follows; $\mu(E)=\mu'(E\cap BV')$ where $E\in\mathscr{B}(L_2)$. Then it is easy to check that I imbeds M' in M and that the question as to whether S' is a proper subset of S is equivalent to the question as to IM' is a proper subset of M [9]. Thus we have that $M'\neq M$, and if μ is the measure generated by the unit mass concentrated in an element $v_0\in L_2-BV'$, then $\mu\in M-M'$. Let

$$\begin{aligned}F(x)&=\int_{L_2}\exp\{i\int_Q v(s,t)\tilde{d}x(s,t)d\mu(v)\\&=\exp\{i\int_Q v_0(s,t)\tilde{d}x(s,t)\}.\end{aligned}$$

Then we have $F\in S^*$, but $F\notin S'$.

Proposition 3.9. *If $F, G\in S^*$ and $F(x)=G(x)$ a.e. on $C_2(Q)$, then $F\cong G$.*

Proof. By definition of S^*, there exist $\mu_1, \mu_2\in M$ such that

$$F(x)\cong\int_{L_2}\exp\{i\int_Q v(s,t)\tilde{d}x(s,t)d\mu_1(v)\tag{3.3}$$

and

$$G(x) \cong \int_{L_2} \exp \{ i \int v(s,t) \tilde{d}x(s,t) \} d\mu_2(v). \tag{3.4}$$

Thus $F(x)$ is almost everywhere equal to both the right hand sides of (3.3) and (3.4). Since $F \in S^* \subset S$, if follows from Proposition 2.1 that μ_1 and μ_2 are identical. Thus $F \cong G$ follows from (3.3) and (3.4).

REMARK 3. If $F \in S^*$, then the values of F on $D_2(Q)$ determine the values of F s-a. e. on $C_2(Q)$; and conversely, the values of F s-a. e. on $C_2(Q)$ determine the values of F everywhere on $D_2(Q)$.

PROPOSITION 3. 10. *If $F \in S$, then there exists $F^* \in S^*$ such that $F^* \approx F$ on $C_2(Q)$.*

Proof. By definition of S, there exists a unique $\mu \in M$ such that

$$F(x) \approx \int_{L_2} \exp \{ i \int_Q v(s,t) \tilde{d}x(s,t) \} d\mu(v)$$

Let

$$F^*(x) = \int_{L_2} \exp \{ i \int_Q v(s,t) \tilde{d}x(s,t) \} d\mu(v).$$

Then F^* is defined whenever the integral exists, and so F^* exists s-a. e. on $C_2(Q)$ and everywhere on $D_2(Q)$. Thus $F^* \in S^*$and $F^* \approx F$.

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