J. Korean Math. Soc. 24(1987), No. 2, pp. 247-255

SASAKIAN SUBMANIFOLDS OF CODIMENSION 2 IN A SASAKIAN MANIFOLD WITH HARMONIC CURVATURE

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0. Introduction

Let N be a Sasakian manifold with the structure tensor (F, G, v). If M is a submanifold immersed in N, M is said to be an *invariant* submanifold provided that the tangent space to M at each point of the submanifold is invariant under the action F. It is well known that an invariant submanifold tangent to the structure vector field v of a Sasakian manifold is also a Sasakian manifold. The study of invariant C-Einstein submanifolds of codimension 2 in a Sasakian manifold, which is called a problem of Nomizu-Smyth, were made by Endo [3], Kon [8], Pak and Oh [10], Yano and Ishihara [13] and so on. One of which, done by Yano and Ishihara [13], asserts that any invariant Einstein submanifold of codimension 2 immersed in a Sasakian manifold of constant curvature is totally geodesic. However, Pak and Oh [10] obtained the same result replacing the Sasakian manifold of constant curvature by that with vanishing C-Bochner curvature tensor.

The main purpose of the present paper is to investigate invariant submanifolds of codimension 2 immersed in an Einstein Sasakian manifold.

1. Invariant submanifolds of codimension 2 in a Sasakian manifold

Let N be a $(2n+1)$-dimensional Sasakian manifold with Sasakian structure (F, G, v) covered by a system of coordinate neighborhoods $\{U ; x^A\}$, where here and in the sequel the indices $A, B, C \ldots$ run over the range $\{1, 2, \ldots, 2n+1\}$. We denote by $F_B{}^A$, G_{BA} and v^A components of the $(1,1)$-tensor F, of the Riemannian metric tensor G and of the

Received May 19, 1987.
This work is supported by KOSEF research grant.

structure vector field v respectively. We then have

$$F_C{}^B F_B{}^A = -\delta_C{}^A + v_C v^A, \quad F_C{}^A v^C = 0, \quad v_A F_C{}^A = 0,$$
$$G_{BA} v^B v^A = 1, \quad F_D{}^B F_C{}^A G_{BA} = G_{DC} - v_D v_C, \tag{1.1}$$

where $v_C = G_{CE} v^E$. Denoting by ∇_A the operator of the covariant differentiation with respect to the fundamental tensor G_{BA}, we have

$$\nabla_C v_B = F_{CB}, \quad \nabla_C F_{EB} = -G_{CE} v_B + G_{CB} v_E. \tag{1.2}$$

From the last equation of (1.2) and the Ricci formula for F_{CB}, we find

$$R_{DCBA} F_E{}^A = -R_{DCE}{}^A F_{AB} - G_{CB} F_{DE} + G_{DB} F_{CE} + G_{CE} F_{DB} - G_{DE} F_{CB}, \tag{1.3}$$

where $R_{DCB}{}^A$ denotes components of the Riemannian curvature tensor of N. Moreover, we have

$$R_{DA} v^A = 2n v_D, \tag{1.4}$$

$$R_{DE} F_B{}^E + R_{BE} F_D{}^E = 0, \tag{1.5}$$

R_{DA} being components of the Ricci tensor of N.

Let M be an invariant submanifold of codimension 2 in N covered by a system of coordinate neighborhoods $\{V ; y^h\}$, where here and in the sequel the indices $h, i, j, \ldots$ run over the range $\{1, 2, \ldots, 2n-1\}$. And let M be immersed isometrically in N by the immersion $i : M \to N$. We represent the immersion i locally by $x^A = x^A(y^h)$ and put $B_j = (B_j{}^A)$ are $(2n-1)$-linearly independent local tangent vector fields of M. We denote by C^A and D^A two mutually orthogonal unit normals to M. Then the induced Riemannian metric g_{ji} on M is given by

$$g_{ji} = G_{CB} B_j{}^C B_i{}^B \tag{1.6}$$

because the immersion is isometric.

Denoting by ∇_j the operator of van der Waerden-Botolotti covariant differentiation formed with g_{ji}, The equations of Gauss and Weingarten for M are respectively obtained:

$$\begin{cases} \nabla_j B_i{}^A = h_{ji} C^A + k_{ji} D^A, \\ \nabla_j C^A = -h_j{}^r B_r{}^A + l_j D^A, \\ \nabla_j D^A = -k_j{}^r B_r{}^A - l_j C^A, \end{cases} \tag{1.7}$$

where $h_j{}^r = h_{ji} g^{ir}$, $k_j{}^r = k_{ji} g^{ir}$ are components of the second fundamental tensors, l_j those of the third fundamental tensor and g^{ji} being contravariant components of g_{ji}. As to the transformations of $B_j{}^A$, C^A and D^A by $F_B{}^A$ we have respectively equations of the form

(1.8) $$F_B{}^A B_i{}^B = f_i{}^j B_j{}^A,$$

(1.9) $$F_B{}^A C^B = D^A, \quad F_B{}^A D^B = -C^A,$$

where we have put $f_{ji}=G(FB_j, B_i)$. The structure vector field v is also represented by

(1.10) $$v^A = p^i B_i{}^A,$$

where $p_i=G(B_i, v)$, p^i being contravariant components of p_i. From the last three equations, it follows that

(1.11) $$\begin{cases} f_j{}^r f_r{}^h = -\delta_j^h + p_j p^h, & p^r f_r{}^h = 0, \\ g_{rs} f_j{}^r f_i{}^s = g_{ji} - p_j p_i, & p^r p_r = 1. \end{cases}$$

Differentiating (1.8) covariantly along N and making use of (1.7), (1.8) and (1.9), we have

(1.12) $$\nabla_j p_i = f_{ji}, \quad \nabla_j f_i{}^h = -g_{ji} p^h + \delta_j^h p_i.$$

Then it is shown that the set (f, g, p) defines a Sasakian structure ([12, 14]). Similarly differentiating (1.10) covariantly and using (1.7) and (1.11), we have

(1.13) $$h_{ji} = k_{jt} f_i{}^t, \quad k_{ji} = -h_{jt} f_i{}^t,$$

which imply

(1.14) $$h_{jr} p^r = 0, \quad k_{jr} p^r = 0,$$
$$h_t{}^t = k_t{}^t = 0.$$

Thus, the submanifold of codimension 2 M is minimal.

From (1.13) we can easily see that

(1.15) $$k_{jr} h_i{}^r + k_{ir} h_j{}^r = 0,$$

(1.16) $$k_{ji}^2 = h_{ji}^2, \quad k_2 = h_2,$$

(1.17) $$k_3 = h_3 = 0,$$

where we have defined $h_{ji}^2 = h_{jr} h_i{}^r$, $k_{ji}^2 = k_{jr} k_i{}^r$, $h_2 = h_{ji} h^{ji}$ and $h_3 = h_{jr} h_s{}^r h^{sj}$.

On the other hand, Gauss, Codazzi and Ricci equations for M are given respectively by

(1.18) $$R_{DCBA} B_k{}^D B_j{}^C B_i{}^B B_h{}^A = R_{kjih} - (h_{kh} h_{ji} - h_{ki} h_{jh} + k_{kh} k_{ji} - k_{ki} k_{jh}),$$

(1.19) $$R_{DCBA} B_k{}^D B_j{}^C B_i{}^B C^A = \nabla_k h_{ji} - \nabla_j h_{ki} - (l_k k_{ji} - l_j k_{ki}),$$

(1.20) $$R_{DCBA} B_k{}^D B_j{}^C B_i{}^B D^A = \nabla_k l_j - \nabla_j l_k + h_{kr} k_j{}^r - h_{jr} k_k{}^r,$$

(1.21) $$R_{DCBA} B_k{}^D B_j{}^C C^B D^A = \nabla_k l_j - \nabla_j l_k + h_{kr} k_j{}^r - h_{jr} k_k{}^r,$$

where R_{kjih} denote covariant components of the Riemannian curvature tensor of M.

Putting $A_{kj}=\nabla_k l_j-\nabla_j l_k$ and utilizing (1.15), the equation (1.21) reduces to

(1.22) $$R_{DCBA}B_k{}^D B_j{}^C C^B D^A=A_{kj}+2h_{kr}k_j{}^r.$$

2. Sasakian submanifolds of codimension 2

Let M be an invariant submanifold of codimension 2 of a Sasakian manifold N. Transvecting (1.18) by g^{ji} and using (1.16), we find

$$R_{DA}B_k{}^D B_h{}^A-R_{DCBA}B_k{}^D B_h{}^A(C^C C^B+D^C D^B)=R_{kh}+2h_{kr}k_h{}^r,$$

where R_{ji} denote components of the Ricci tensor of M. Transvecting $f_j{}^h$, we get

(2.1) $$R_{kr}f_j{}^r-2h_{kr}k_j{}^r=R_{DA}B_k{}^D f_j{}^r B_r{}^A-R_{DCBA}F_E{}^A B_j{}^E B_k{}^D(C^C C^B+D^C D^B)$$

On the other hand, from (1.3) we have

$$R_{DCBA}F_E{}^A B_k{}^D B_j{}^E(C^C C^B+D^C D^B)$$
$$=R_{DCEA}F_B{}^A B_k{}^D B_j{}^E C^C C^B+R_{DCEA}F_B{}^A B_k{}^D B_j{}^E D^C D^B+2F_{ED}B_k{}^D B_j{}^E.$$

Thus, using (1.8), (1.9) and (1.22), it follows that

$$R_{DCBA}F_E{}^A B_k{}^D B_j{}^E(C^C C^B+D^C D^B)=A_{kj}+2h_{kr}k_j{}^r-2f_{kj}.$$

Therefore (2.1) turns out to be

$$R_{DA}B_k{}^D f_j{}^r B_r{}^A=R_{kr}f_j{}^r+A_{kj}-2f_{kj}.$$

Transvecting $f_i{}^j$ and taking account of (1.11), we get

$$R_{DA}B_k{}^D(-\delta_i^r+p_i p^r)B_r{}^A=R_{kr}(-\delta_i^r+p_i p^r)+A_{kr}f_i{}^r-2(g_{ki}-p_k p_i),$$

which togther with (1.4) and (1.10) yields

$$R_{DA}B_j{}^D B_i{}^A=R_{ji}+2g_{ji}-A_{jr}f_i{}^r.$$

LEMMA 1. *Let N be a $(2n+1)$-dimensional Sasakian manifold with harmonic curvature.*

Proof. Differentiating (1.4) covariantly along N and using (1.1), we find

$$(\nabla_C R_{DA})v^A+R_{DA}F_C{}^A=2nF_{CD}.$$

Since N has harmonic curvature, namely, $\nabla_C R_{DA}-\nabla_D R_{CA}=0$, it follows that

$$R_{DA}F_C{}^A-R_{CA}F_D{}^A=4nF_{CD},$$

which together with (1. 5) yields

(2. 3) $$R_{DA}F_C{}^A=2nF_{CD}.$$

Transvecting (2. 3) by $F_B{}^C$ and making use of (1. 1) and (1. 4), we see that N is Einstein. This completes the proof.

Since M is also Sasakian manifold, we see that

(2. 4) $$f_j{}^rR_{ir}=f^{lk}R_{ljik}+(2n-3)f_{ji}.$$

Thus, if M is of harmonic curvature, then, by Lemma 1, M is Einstein, i. e. $R_{ji}=2(n-1)g_{ji}$. Therefore (2. 4) turns out to be

$$f^{lk}R_{ljik}=f_{ji}.$$

Transvecting (1. 22) with p^j and using (1. 10) and (1. 14), we find

$$R_{DCBA}B_k{}^Dv^CC^BD^A=A_{kj}p^j,$$

However, we have $R_{DCBA}v^A=v_DG_{CB}-v_CG_{DB}$, which is a direct consequence of (1. 2), we get

$$R_{BADC}B_k{}^Dv^CC^BD^A=0.$$

Therefore we have

(2. 6) $$A_{jr}p^r=0.$$

LEMMA 2. *Let M be an invariant submanifold of codimension 2 in a Sasakian manifold. Then we have*

(2. 7) $$\nabla^l\nabla_jh_{il}=(\nabla_l\nabla_jk_{ik})f^{lk}+(2n-3)h_{ji}.$$

Proof. From (1. 14), we have respectively

$$(\nabla_kk_{jr})p^r+k_{jr}f_k{}^r=0,\quad (\nabla_kh_{jr})p^r+h_{jr}f_k{}^r=0$$

because of (1. 12). By using (1. 13), it follows that

(2. 8) $$(\nabla_kk_{jr})p^r=-h_{kj},\quad (\nabla_kh_{jr})p^r=k_{jk}.$$

Differentating (1. 13) covariantly along M and taking account of (1. 12), we find respectively

(2. 9) $$\nabla_kh_{ji}=(\nabla_kk_{jr})f_i{}^r+k_{jk}p_i,\quad \nabla_kk_{ji}=-(\nabla_kh_{jr})f_i{}^r-h_{jk}p_i.$$

Differentiating (2. 9) covariantly and using (1. 12) we obtain

$$\nabla_l\nabla_kh_{ji}=(\nabla_l\nabla_kk_{jr})f_i{}^r+(\nabla_kk_{jr})(-g_{li}p^r+g_{lr}p_i)+(\nabla_lk_{jk})p_i+k_{kj}f_{li},$$

or, making use of (2. 8),

$$\nabla_l\nabla_kh_{ji}=(\nabla_l\nabla_kk_{jr})f_i{}^r+g_{li}h_{kj}+(\nabla_kk_{jl})p_i+(\nabla_lk_{jk})p_i+k_{kj}f_{li}.$$

If we transvect g^{li} and use (2. 8), then we obtain the equation (2. 7) which completes the proof.

3. Invariant submanifolds of an Einstein manifold

Suppose that N is a $(2n+1)$-dimensional Einstein manifold. Then we have $R_{DA}=2nG_{DA}$. It means that

$$R_{DA}B_j{}^D B_i{}^A=2ng_{ji}.$$

Thus (2.2) reduces to

$$R_{ji}=2(n-1)g_{ji}+A_{jr}f_i{}^r. \tag{3.1}$$

Substituting (3.1) into (2.4) and using (2.6), we find

$$f^{lk}R_{ljik}=f_{ji}+A_{ji}.$$

By the properties of the Sasakian structure, it follows that

$$f^{lk}R_{lkji}=-2(f_{ji}+A_{ji}).$$

On the other hand, from the Ricci identity for k_{ji}, we obtain

$$\nabla_l\nabla_k k_{ji}-\nabla_k\nabla_l k_{ji}=-R_{lkjr}k_i{}^r-R_{lkir}k_j{}^r.$$

Transvecting the above equation with f^{lk} and taking account of (1.13) and (3.2), it reduces to

$$f^{lk}\nabla_l\nabla_k k_{ji}=2h_{ji}+A_{jr}k_i{}^r+A_{ir}k_j{}^r. \tag{3.3}$$

Thus, it follows that

$$h^{ji}f^{lk}\nabla_l\nabla_k k_{ji}=2h_2+A_{jr}k_i{}^r h^{ji}. \tag{3.4}$$

PROPOSITION 3. *Let N be a Sasakian with harmonic curvature. If M is an invariant submanifold of codimension 2 in N, then the following assertions are true*:

(1) *M is an Einstein manifold if and only if the normal connection of M is flat.*

(2) *M is C-Einstein manifold if and only if $A_{ji}=-bf_{ji}$ for some constant b.*

Proof. (1) From Lemma 1 we have (3.1). Thus it is easily seen that $R_{ji}=2(n-1)g_{ji}$ is equivalent to $A_{ji}=0$ by taking account of (2.6).

(2) "If part" is evidently true because of (3.1).

Suppose that M is of C-Einstein, i.e. $R_{ji}=ag_{ji}+bp_jp_i$. Then from (3.1) we have

$$-b(g_{ji}-p_jp_i)=A_{jr}f_i{}^r.$$

Differentiating the above equation covariantly and (2.6), we get

$$b(f_{kj}+f_{ki}p_j)=(\nabla_k A_{jr})f_i{}^r+A_{jk}p_i.$$

Transvecting with p^i, we find $bf_{kj}=A_{jk}$. Thus we complete the proof.

THEOREM 4. *Let N be a $(2n+1)$-dimensional Sasakian manifold with harmonic curvature and M be a compact C-Einstein submanifold of codimension 2 in N. If the second fundamental form is of Codazzi type and the scalar curvature of M is non-negative, then M is totally geodesic.*

Proof. Since h_{ji} is of Codazzi type, Lemma 2 tells us that

$$\Delta h_{ji}=(\nabla_j\nabla_k k_{ji})f^{jk}+(2n-3)h_{ji}, \tag{3.5}$$

where $\Delta=\nabla^i\nabla_i$ denotes the operator of Laplacian, which and (3.4) imply

$$h^{ji}\Delta h_{ji}=(2n-1)h_2+A_{jr}k_i{}^r h^{ji}. \tag{3.6}$$

Since M is C-Einstein manifold, by Proposition 3, we see that $A_{jr}=-bf_{jr}$. Thus (3.6) reduces to

$$h^{ji}\Delta h_{ji}=(2n-1-b)h_2 \tag{3.7}$$

because of (1.13). From (3.1) we have

$$R=2(n-1)(2n-1-b)\geqq 0$$

because the scalar curvature of M is non-negative is assumed. Therefore (3.7) means

$$h^{ji}\Delta h_{ji}=\frac{R}{2(n-1)}h_2.$$

Therefore we have the following identity:

$$\frac{1}{2}\Delta h_2=\frac{R}{2(n-1)}h_2+\|\nabla_k h_{ji}\|^2\geqq 0. \tag{3.8}$$

Since M is compact, we see, according to the Green's Theorem, that $h_2=k_2=0$ because of (3.5). Thus M is totally geodesic because of (1.16).

Now, we prove the following theorem:

THEOREM 5. *Let M be a compact invariant submanifold of codimension 2 with harmonic curvature in a Sasakian manifold. If the second fundamental form is of Codazzi type, then M is totally geodesic.*

Proof. Since M is of also a Sasakian manifold with harmonic curvature, by Lemma 1, it follows that M is Einstein, that is,

$$R_{ji}=2(n-1)g_{ji}.$$

Thus (2.5) is valid. Hence (3.3) becomes

$$f^{lk}\nabla_k\nabla_l k_{ji}=2h_{ji}.$$

Therefore (3.5) reduces to

(3.9) $$\Delta h_{ji}=(2n-1)h_{ji}$$

because h_{ji} is of Codazzi type. M being compact, combining (3.8) and (3.9), it is clear that $h_{ji}=k_{ji}=0$. Thus M is totally geodesic.

REMARK 1. If we replace the compactness condition in Theorem 4 and Theorem 5 by h_2=constant. Then we obtain the same result that M is totally geodesic.

REMARK 2. Let N be a Sasakian space form, then it is clear that h_2=constant and h_{ji} is of Codazzi type. Thus it follows that if M is an invariant submanifold of codimension 2 in a Sasakian space form $N(c)$, then M is totally geodesic (cf. [10]).

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