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BEST APPROXIMATION BY CERTAIN COMPACT OPERATORS ON $L^p(\{-1,1\}^N)$

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1. Introduction

Let M be a closed subspace of a Banach space X. An element x in X is said to have a best approximation in M if there exists an element y in M such that $\|x-y\|=\inf\{\|x-z\| : z\in M\}$. M is called proximinial in X if every element in X has a best approximation in M. Obviously, every finite dimensional subspace of a Banach space is proximinal, and it is known that every subspace of a Banach space X is proximinal exactly when X is reflexive [18]. If a closed subspace M of X is a semi M-ideal [15] or has the 1/2 ball property [22], M is proximinal in X. If M is an M-ideal in X, then for every $x\in X\backslash M$ the set of all best approximations in M of x is so large that it algebraically spans M [13].

Of particular interest is finding Banach spaces X and Y for which $K(X,Y)$, the space of the compact linear operators from X to Y, is proximinal in $L(X,Y)$, the space of the bounded linear operators from X to Y. If $X=Y$ we simply write $L(X)$ (resp. $K(X)$) for $L(X,X)$ (resp. $K(X,X)$). There are several examples of Banach spaces X and Y for which $K(X,Y)$ is proximinal (resp. an M-ideal) in $L(X,Y)$ [2, 12, 14, 17, 22] (resp. [4, 5, 6, 14, 16, 20]). $K(c_0)$, $K(l^1)$ and $K(l^p, l^q)$ for $1<p, q<\infty$ are proximinal in corresponding space of operators [16, 22]. If $X=l^\infty$, $L^1(0,1)$ or $L^\infty(0,1)$, $K(X)$ is not proximinal in $L(X)$ [7]. It is only recent that Benyamini and Lin [3] proved that $K(X)$ is not proximinal in $L(X)$ for $X=L^p(0,1)$, $1<p\neq 2<\infty$. However, it was known earlier that certain integral operators in $L(X)$ have best approximations in $K(X)$ which are also integral operators if $X=L^p(\Omega,\mu)$ where (Ω,μ) is a finite measure spece [21].

In this paper, we consider Lebesgue space $X=L^p(\{-1, 1\}^N, \lambda)$,

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$1\leqslant p\neq 2<\infty$, where λ is the Haar measure on compact group $\{-1,1\}^N$ which is a countable product of copies of multiplicative group $\{-1,1\}$. In Theorem 3.4, we show that certain convolution operators in $L(X)$ have best approximations in the left ideal $F(X)$ of the compact operators on X which annihilate constant functions.

2. Preliminaries

This section contains some background material which will be needed in the proof of the main theorem in section 3.

Let $(\Omega_r, \mathcal{A}_r, \mu_r)$ denote a probability space for each r in an infinite index set Γ and let $(\Omega_A, \mathcal{A}_A, \mu_A)$ denote the product measure space of $\{(\Omega_r, \mathcal{A}_r, \mu_r);\ r\in A\}$ for a nonempty subset A of Γ. We simply write $(\Omega, \mathcal{A}, \mu)$ for $(\Omega_\Gamma, \mathcal{A}_\Gamma, \mu_\Gamma)$. An element t of Ω can be written as $t=(t_r)$, where $t_r\in\Omega_r$ is the value of t at r. For a proper subset A of Γ, we regard $(\Omega, \mathcal{A}, \mu)$ as $(\Omega_A\times\Omega_{A'},\ \mathcal{A}_A\times\mathcal{A}_{A'},\ \mu_A\times\mu_{A'})$, where $A'=\Gamma\backslash A$, and hence an element t in Ω can be written as $t=(r,s)$ with $r\in\Omega_A$ and $s\in\Omega_{A'}$ The following is proved in [10, p. 437].

Theorem 2.1. *Let A be a nonempty finite subset of an infinite index set Γ. For $1\leqslant p<\infty$ and $f\in L^p(\Omega,\mu)$, we define Q_Af by*

$$(Q_Af)(r,s)=\int_{\Omega_{A'}} f(r,w)\,d\mu_{A'}(w).$$

Then Q_Af is in $L^p(\Omega,\mu)$, $\|Q_Af\|\leqslant\|f\|$ and $\lim_A\|Q_Af-f\|=0$.

Remark. Observe that if $f\in L^p(\Omega,\mu)$, Q_Af can be defined and $\|Q_Af\|\leqslant\|f\|$ for an infinite subset A of Γ or for $p=\infty$. Since Q_A fixes constant functions, Q_A is a norm one projection on $L^p(\Omega,\mu)$ for $1\leqslant p\leqslant\infty$ and $\phi\neq A\subseteq\Gamma$.

Suppose λ is a left Haar measure on a locally compact (topological) group G. If $f\in L^p(G,\lambda)$ $(1\leqslant p\leqslant\infty)$ and μ is a complex regular Borel measure on G, then the convolution $\mu*f$ of μ and f is defined as the following theorem shows.

Theorem 2.2 [9, p. 292] *Let λ (resp. μ) be a left Haar (resp. complex regular Borel) measure on a locally compact group G and let $f\in L^p(G,\lambda)$ $(1\leqslant p\leqslant\infty)$. Then the integral*

$$(\mu*f)(x)=\int_G f(y^{-1}x)\,d\mu(y)$$

exists and is finite for all $x\in G\cap N'$, *where* N *is a* λ*-null set if* $1\leqslant p<\infty$ *and* N *is a locally* λ*-null set* (*that is,* $N\cap K$ *is* λ*-null set for any compact subset* K *of* G) *if* $p=\infty$. *Defining* $(\mu*f)(x)=0$ *where the above integral is not defined, we get a function* $\mu*f$ *in* $L^p(G, \lambda)$ *such that*

$$\|\mu*f\|_p\leqslant\|\mu\|\ \|f\|_p.$$

3. Best approximation by certain compact operators

Let multiplicative group $\{-1, 1\}$ be endowed with the discrete topology, and for each n in N, the set of all natural numbers, let $(\{-1, 1\}, \mathfrak{F}_n, \lambda_n)$ be the probability space such that $\lambda_n(\{-1\})=\lambda_n(\{1\})=1/2$, where $\mathfrak{F}_n=\{\{-1, 1\}, \phi, \{-1\}, \{1\}\}$. If A is a nonempty subset of N, $\{\{-1, 1\}_A, \mathfrak{F}_A, \lambda_A\}$ will denote the product measure space of $\{(\{-1, 1\}, \mathfrak{F}_j, \lambda_j); j\in A\}$ as in section 2. For notational convenience, if $A=\{1, 2, 3, \ldots, n\}$ we will write $(\{-1, 1\}^n, \mathfrak{F}^n, \lambda^n)$, $(\{-1, 1\}^{(n)}, \mathfrak{F}^{(n)}, \lambda^{(n)})$ and $(\{-1, 1\}^N, \mathfrak{F}, \lambda)$ for $(\{-1, 1\}_A, \mathfrak{F}_A, \lambda_A)$, $(\{-1, 1\}_{A'}, \mathfrak{F}_{A'}, \lambda_{A'})$ and $(\{-1, 1\}_N, \mathfrak{F}_N, \lambda_N)$, respectively. In what follows, $L^p(\{-1, 1\}_A)$ will always denote $L^p(\{-1, 1\}_A, \lambda_A)$. We can easily see that each λ_A is the (normalized) Haar measure on the compact group $\{-1, 1\}_A$. Of course, the group multiplication in $\{-1, 1\}_A$ is defined coordinatewise and the topology for $\{-1, 1\}_A$ is the product topology. If $1\leqslant p<\infty$, for each $f\in L^p(\{-1, 1\}_A)$ the assignment $f(r)\longrightarrow \tilde{f}(r, s)=f(r)$ defines a linear isometry from $L^p(\{-1, 1\}_A)$ into $L^p(\{-1, 1\}^N)$. Thus we can view $L^p(\{-1, 1\}_A)$ as a subspace of $L^p(\{-1, 1\}^N)$.

For each $i\in N$, let R_i be the i-th coordinate projection on $\{-1, 1\}^N$, that is, $R_i(t)=t_i$ for $t=(t_n)_{n=1}^{\infty}\in\{-1, 1\}^N$. $\{R_i\}_{n=1}^{\infty}$ is a (stochastically) independent family of random variables with

$$\int_{\{-1, 1\}^N} R_i(t)\,d\lambda(t)=0$$

for $i\in N$. For a finite subset C of N, we define

$$W_C=\prod_{i\in C} R_i.$$

Here W_ϕ is the constant 1 function. Each W_C (in particular, R_i) is a character of the group $\{-1, 1\}^N$. A character of a locally compact group G is a continuous homomorphism of G into the multiplicative group of complex numbers of absolute value one.

By a slight abuse of notations, we will still write R_i for the i-th coordinate projection on $\{-1, 1\}^n$ for $i \leqslant n$. Again, $R_1, R_2, \ldots, R_n$ are independent random variables on the probability space $(\{-1, 1\}^n, \mathcal{F}^n, \lambda^n)$ with mean zero, that is,

$$\int_{\{-1,1\}^n} R_i(t)\, d\lambda^n(t) = 0 \quad (i=1, 2, \ldots, n)$$

and W_C is a character of $\{-1, 1\}^n$ if $C \subseteq \{1, 2, \ldots, n\}$.

THEOREM 3.1. *For* $1 \leqslant p < \infty$, *the algebraic span of* $\{W_C : C \subseteq N$ *and* C *is finite*$\}$ *is dense in* $L^p(\{-1, 1\}^N)$.

Proof. For a fixed $n \in N$, the group $\{-1, 1\}^n$ and the set $W_C : C \subseteq \{1, 2, \ldots, n\}\}$ have the same cardinality 2^n. Since $R_1, R_2, \ldots, R_n$ are independent and each R_i has mean zero on the probability space $(\{-1, 1\}^n, \mathcal{F}^n, \lambda^n)$ for any two distinct subsets of $\{1, 2, \ldots, n\}$ we have

$$\int_{\{-1,1\}^n} W_B W_C\, d\lambda^n = 0.$$

Thus $\{W_C : C \subseteq \{1, 2, \ldots, n\}\}$ is an orthonormal family with cardinality 2^n in the 2^n-dimensional Hilbert space $L^2(\{-1, 1\}^n)$ and forms a basis for $L^2(\{-1, 1\}^n)$. It also spans $L^p(\{-1, 1\}^n)$, since $L^p(\{-1, 1\}^n) = L^2(\{-1, 1\}^n)$ as a set for $1 \leqslant p < \infty$. On the other hand, if we let Q_n be Q_A defined in Theorem 2.1, where $A = \{1, 2, \ldots, n\}$, then by Theorem 2.1, for any $f \in L^p(\{-1, 1\}^N)$, $\lim \|Q_n f - f\|_p = 0$ and $Q_n f \in L^p(\{-1, 1\}^n) \subseteq L^p(\{-1, 1\}^N)$. This completes the proof.

COROLLARY 3.2 *The family* $\{W_C : C \subseteq N$ *and* C *is finite*$\}$ *is a complete orthonormal system in* $L^2(\{-1, 1\}^N)$.

REMARKS. 1. It is well known in harmonic analysis that a finite abelian group G is isomorphic to its character group[9 : p. 367]. Since $\mathcal{W} = \{W_C : C \subseteq \{1, 2, \ldots, n\}\}$ and $\{-1, 1\}^n$ have the same cardinality, $\mathcal{W}$ is the character group of $\{-1, 1\}^n$.

2. Observe that the proof of Theorem 3.1 actually shows that, for any subset A of N, the algebraic span of $\{W_C : C \subseteq A$ and C is finite$\}$ is dense in $L^p(\{-1, 1\}_A)$ for $1 \leqslant p < \infty$.

Let $(\{-1, 1\}^N, \mathcal{F}, \mu)$ be the product space of a sequence $\{(\{-1, 1\}, \mathcal{F}_n, \mu_n)\}_{n=1}^{\infty}$ of probability spaces, where $\mathcal{F}_n = \{\{-1, 1\}, \phi, \{-1\}, \{1\}\}$. In what follows, measure μ always represents this measure. Clearly μ is a regular Borel probability measure on the compact group $\{-1, 1\}^N$. Hence, by Theorem 2.2 μ induces a bounded linear operator T_μ on $L^p(\{-1, 1\}^N)$

by $T_\mu f=\mu * f$ for $f\in L^p$ $(\{-1,1\}^N)$, where

$$(\mu * f)(x)=\int_{\{-1,1\}^N} f(y^{-1}x)d\mu(y).$$

THEOREM 3.3. *For each nonempty subset A of N, T_μ restricted to $L^p(\{-1,1\}_A)$ is an $L^p(\{-1,1\}_A)$-operator, more precisely, $T_\mu|_{L^p(\{-1,1\}_A)} = T_{\mu_A}: L^p(\{-1,1\}_A)\longrightarrow L^p(\{-1,1\}_A)$ and T_{μ_A} has norm one.*

Proof. For each $i\in N$ and $x\in\{-1,1\}^N$,

$$\begin{aligned}(T_\mu R_i)(x) &= (\mu * R_i)(x)\\ &=\int_{\{-1,1\}^N} R_i(y^{-1}x)d\mu(y)\\ &=\int_{\{-1,1\}^N} R_i(x)R_i(y)d\mu(y)\\ &=R_i(x)\int_{\{-1,1\}^N} R_i(y)d\mu(y)\\ &=(1-2a_i)R_i(x), \quad \text{where } \mu_i(\{-1\})=a_i, 0\leqslant a_i\leqslant 1.\end{aligned}$$

Since $R_1, R_2, \ldots, R_n, \ldots$ are independent, a straight forward computation shows that for each nonempty finite subset C of A,

$$T_\mu W_C=(\prod_{i\in C}(1-2a_i))W_C.$$

Similarly if we view W_C as a character of $\{-1,1\}_A$, the same computation gives

$$T_{\mu_A} W_C=(\prod_{i\in C}(1-2a_i))W_C.$$

This proves the first part of the theorem, since algebraic span of $\{W_C : C\subseteq A\}$ is dense in $L^p(\{-,1\}_A)$.

By Theorem 2.3, $\|T_{\mu_A}\|\leqslant 1$. However, if f is the constant 1 function, $T_{\mu_A}f=\mu_A(\{-1,1\}_A)=1=\|f\|_p$. Hence $\|T_{\mu_A}\|=1$ for all nonempty subset of N.

THEOREM 3.4. *Operator T_μ on $X=L^p(\{-1,1\}^N)$, $1\leqslant p<\infty$ has a best approximation in the closed left ideal $F(X)$ of all compact operators on X which annihilate constant functions.*

Proof. It is clear that $F(X)$ is a closed left ideal in $L(X)$. Let Q_n be the projection on X defined in Theorem 3.1. Using Fubini's theorem and Riesz Representation Theorem on $L^p(\{-1,1\}^N)$, we can easily see that Q_A is self-adjoint as an operator on $L^p(\{-1,1\}^N)$.

Fix $K\in F(X)$. Since for each $f\in X$, $\|f-Q_nf\|\to 0$ as $n\to\infty$ by

Theorem 2.1 and K is compact, we have

$$\|K-Q_nK\|=\|(I-Q_n)K\|\to 0 \text{ as } n\to\infty.$$

Since the adjoint K^* of K is also compact and Q_n is self-adjoint,

$$\|K-KQ_n\|=\|K^*-Q_nK^*\|\longrightarrow 0 \text{ as } n\to\infty.$$

On the other hand KQ_n restricted to $L^p(\{-1,1\}^{(n)})$ is zero operator, since $Q_nW_C=0$ for $\phi\neq C\subseteq\{n+1,n+2,\dots\}$, $KQ_n(W_\phi)=K(W_\phi)=0$ and algebraic span of $\{W_C : C\subseteq\{n+1,n+2,\dots\}\}$ is dense in $L^p(\{-1,1\}^{(n)})$. Hence,

$$\begin{aligned}\|T_\mu-K\|&=\lim_{n\to\infty}\|T_\mu-KQ_n\|\\ &\geqslant\lim_n\inf\|(T_\mu-KQ_n)|_{L^p(\{-1,1\}^{(n)})}\|\\ &=\lim_n\inf\|T_\mu|_{L^p(\{-1,1\}^{(n)})}\|\\ &=1 \text{ by Theorem 3.3.}\end{aligned}$$

Theorefore, $\|T_\mu-0\|=1=\inf\{\|T_\mu-K\| : K\in F(X)\}$ and T_μ has zero operator as a best approximation in $F(X)$.

Here, it is natural to ask whether the space of all compact operators on $L^p(\{-1,1\}^N)$ is a proximinal subspace or an M-ideal in the space of the bounded linear operators on $L^p(\{-1,1\}^N)$.

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