J. Korean Math. Soc. 24(1987), No. 2, pp. 217-237

APPLICATION OF CONVOLUTION OPERATORS TO SOME PROBLEMS IN GEOMETRIC FUNCTION THEORY

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1. Introduction

Let A denote the class of analytic functions $f(z)$ in the unit disk $E=\{z: |z|<1\}$ with $f(0)=0$, $f'(0)=1$. We denote by S the subclass of A consisting of univalent functions. Let K, S^*, C and S_p be the standard subclasses of S consisting of the convex, starlike, close-to-convex, and spirallike functions, respectively.

Let f and g be in A with $f(z)=\sum_{n=0}^{\infty} a_n z^n$ and $g(z)=\sum_{n=0}^{\infty} b_n z^n$. We define the convolution operator $\Gamma: A \longrightarrow A$ by $\Gamma(g)=f*g$ for given $f\in A$, where $(f*g)(z)=\sum_{n=0}^{\infty} a_n b_n z^n$. Let h be a convex function in E. It is known that if f is in K, S^* or C, then $h*f$ is in K, S^* or C, respectively.

Let Γ_i, $0\le i\le 4$, be the linear operator defined on A by the equation below.

$$\Gamma_0 f(z)=zf'(z),\quad \Gamma_1 f(z)=[f(z)+zf'(z)]/2$$

$$\Gamma_2 f(z)=\int_0^z \frac{f(\zeta)-f(0)}{\zeta}d\zeta,\quad \Gamma_3 f(z)=\frac{2}{z}\int_0^z f(\zeta)d\zeta$$

$$\Gamma_4 f(z)=\int_0^z \frac{f(\zeta)-f(x\zeta)}{\zeta-x\zeta}d\zeta,\quad |x|\le 1, x\neq 1.$$

Each of these operators can be written as a convolution operator given by $\Gamma_i f=h_i*f$, $0\le i\le 4$ where

$$h_0(z)=\sum_{n=1}^{\infty} nz^n=\frac{z}{(1-z)^2},\quad h_1(z)=\sum_{n=1}^{\infty}\frac{n+1}{2}z^n=\frac{z-z^2/2}{(1-z)^2}$$

$$h_2(z)=\sum_{n=1}^{\infty}\frac{1}{n}z^n=-log(1-z),\quad h_3(z)=\sum_{n=1}^{\infty}\frac{2}{n+1}z^n=\frac{-2[z+log(1-z)]}{z}$$

Received April 24, 1987.
Supported by a grant from the Korea Science and Engineering Foundation, 1985-87.

$$h_4(z)=\sum_{n=1}^{\infty}\frac{1-x^n}{(1-x)n}z^n=\frac{1}{1-x}log\left[\frac{1-xz}{1-z}\right],\quad |x|\leq 1,\quad x\neq 1.$$

For a given compact subclass X of A let $r_S[X]$ denote the minimum radius of univalence over all functions f in X. We use the corresponding notation for the other subclasses of S. For example $r_{S*}[X]$ denotes the minimum radius of starlikeness over all functions f in X. Robinson observed that for any f in S, the derivative of $\Gamma_1 f(z)$ does not vanish for $|z|<1/2$. He also noted that for the standard Köebe function $k(z)=z(1-z)^{-2}$, $r_S[\Gamma_1(k)]=1/2$. He conjectured that $r_X[\Gamma_1(X)]=1/2$ for $X=S$. Although 1/2 has been verified to be the correct radius when $X$is replaced by many of the subclasses of S, Robinson's lower bound of. 38 for r_S $[\Gamma_1(S)]$ has not been improved until to 1978. A straightforward argument using convolution techniques and Krzyz's result determining $r_C(S)$ can be used to show that $r_S[\Gamma_1(S)]>.417$. Barnard [2] has proved in 1978 that

$$.49<r_S[\Gamma_1(S)]\leq .50.$$

The classical results of Alexander show that

$$r_{S*}[\Gamma_0(S)]=r_{S*}[\Gamma_0(S^*)]=2-\sqrt{3},$$

and Livingston [14] proved that

$$r_K[\Gamma_1(K)]=r_{S*}[\Gamma_1(S^*)]=r_C[\Gamma_1(C)]=1/2.$$

Generalizations of these results have been given by Libera and Livinston in [13] and by Bernardi in [4]. It has been shown by Causey and others that $r_C[\Gamma_2(C)]=1$. Libera [12] showed that

$$r_K[\Gamma_3(K)]=r_{S*}[\Gamma_3(S^*)]=r_C[\Gamma_3(C)]=1$$

and these results have been generalized by Bernardi in [3]. Pommerenke [18] has shown that $r_C[\Gamma_4(C)]=1$.

It is not difficult to find the radius of convexity of each of the functions h_i, $0\leq i\leq 4$, previously defined, that is, $r_K[h_0]=2-\sqrt{3}$, $r_K[h_1]=1/2$ and

$$r_K[h_2]=r_K[h_3]=r_K[h_4]=1.$$

It is known that if f is in K, S^*, or C then $\Gamma_i f=h_i * f$ is convex, starlike or close to convex, respectively, up to $r_K[h_i]$ for each i, $0\leqslant i\leqslant 4$.

For a given α, $\alpha<1$, a function $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ is said to be in the class of functions starlike of order α, denoted by $\mathcal{S}_\alpha$, if

Re $z'f(z)/f(z)>\alpha$ $(z\in E)$. If $0\leqslant\alpha<1$, then $\mathcal{S}_\alpha\subset S$. The function

$$s_\alpha(z)=\frac{z}{(1-z)^{2(1-\alpha)}}$$

is the well known extremal function for the class $\mathcal{S}_\alpha$. Letting

$$C(\alpha,n)=\frac{\prod_{k=2}^{n}(k-2\alpha)}{(n-1)!}\quad(n=2,3,\ldots),\tag{1.1}$$

s_α can be written in the form $s_\alpha(z)=z+\sum_{n=2}^{\infty}C(\alpha,n)z^n$

A function f in A is said to be in the class of functions convex of order α, denoted by $\mathcal{K}_\alpha$, if Re $\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha$ $(z\in E)$. For $0\leqslant\alpha<1$, $\mathcal{K}_\alpha\subset S$. Also it is well known that $f\in\mathcal{K}_\alpha$ if and only if $s_0*f=zf'(z)\in\mathcal{S}_\alpha$.

A function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in A$ is said to be in the class of functions prestarlike of order α $(\alpha<1)$, denoted by $\mathcal{R}_\alpha$, if $f*s_\alpha\in\mathcal{S}_\alpha$. It is shown in [31] that $\mathcal{R}_\alpha\subset S$ if and only if $\alpha\leqslant1/2$. For convenience, we introduce the notation $s_\alpha^{-1}(z)$ for the function normalized by $s_\alpha^{-1}(0)=0$ with $(s_\alpha^{-1}*s_\alpha)(z)=z/(1-z)$. Thus

$$s_\alpha^{-1}(z)=z+\sum_{n=2}^{\infty}(1/C(\alpha,n)z^n,$$

where the $C(\alpha,n)$ are given by (1.1). With this notation, we observe that $s_\alpha^{-1}*g\in\mathcal{R}_\alpha$ for all $g\in\mathcal{S}_\alpha$.

The family $\mathcal{R}_\alpha$ was introduced in [33] where it was shown that

$$\mathcal{R}_\alpha\subset\mathcal{R}_\beta\tag{1.2}$$

for $\alpha<\beta<1$, and

$$f*g\in\mathcal{R}_\alpha\tag{1.3}$$

for $f,g\in\mathcal{R}_\alpha$.

In this paper, we are able to generalize previously known results and obtain some results including a verification of Robinson's 1/2 conjecture in the case of spirallike functions in Section 2. In Section 3, we give a generalization of Bernardi's result. Throughout Bernardi's paper [4], he considered c a positive integer. We shall consider c a complex number with $c\neq-1$. Moreover, we obtain a generalization of Burdick, keogh and Merkes results by using convolution technique.

In Section 4, we are concerned with two generalizations of the class $\mathcal{R}_\alpha$. We define the new classes $\mathcal{R}(\alpha, \beta)$ and $Q(\alpha, \beta)$ and discuss their relationship to each other and to the class $\mathcal{R}_\alpha$. We obtain the necessary and sufficient conditions for inclusion in the new classes.

2. Preparatory lemmas and its applications

The following simple lemma [26] is applied usefully in this paper.

LEMMA 1. *Suppose ϕ, g are analytic in $|z|<1$ and that*

$$\phi(z)*\frac{1+xz}{1-yz}g(z)\neq 0$$

for $|x|=|y|=1$, $|z|<1$. Then if $F\in A$ and Re $F>0$,

$$\mathrm{Re}\frac{\phi*gF}{\phi*g}>0. \tag{2.1}$$

Proof. When $x=-y$ we obtain $\phi*g\neq 0$, and by simple manipulations we obtain for $|x|\leq 1$, $|y|=1$

$$\frac{\phi(z)*\frac{1+yz}{1-yz}g(z)}{\phi(z)*g(z)}\neq\frac{x\bar{y}-1}{x\bar{y}+1}$$

and hence (2.1) holds for $F(z)=(1+yz)/(1-yz)$. If Re $F>0$ we have by Herglotz's theorem

$$F(z)=\int_{|y|=1}\frac{1+yz}{1-yz}d\mu(y)+ic$$

where μ is a positive measure on the unit circle $|y|=1$ and c is a real constant. Hence

$$\mathrm{Re}\frac{\phi*gF}{\phi*g}=\int_{|y|=1}\mathrm{Re}\frac{\phi(z)*\frac{1+yz}{1-yz}g(z)}{\phi(z)*g(z)}\,d\mu>0.$$

We shall need the slightly more general version of the key lemma as follows:

LEMMA 2. *Let ϕ and g be analytic in $|z|<1$ with $\phi(0)=g(0)=0$ and $\phi'(0)g'(0)\neq 0$. Suppose that for each α ($|\alpha|=1$) and σ ($|\sigma|=1$) we have*

$$\left[\phi*\left(\frac{1+\alpha\sigma z}{1-\sigma z}\right)g\right](z)\neq 0 \quad \text{on } 0<|z|<r\leqslant 1. \tag{2.2}$$

Then for each F in A the image of $|z|<r$ under $(\phi*Fg)/(\phi*g)$ is a subset of the convex hull of $F(E)$.

Proof. Since $\phi(z)*[(1+\alpha\sigma z)/(1-\sigma z)]g(z)\neq 0$ for $0<|z|<r\leqslant 1$ is equivalent to $\phi(rz)*[(1+\alpha\sigma z)/(1-\sigma z)]g(z)\neq 0$ for $0<|z|<1$ we can assume $r=1$. By Lemma 1, if F has positive real part and (2.2) is satisfied then $\mathrm{Re}\{[(\phi*gF)/(\phi*g)](z)\}>0$ for z in E. For arbitrary F in A the convex hull of $F(E)$ is defined to be the total intersection of all half planes containing $F(E)$. If we denote by $\overline{F(E)}$ the closure of $F(E)$ then a line of support l of $F(E)$ is the boundary of a half plane containing $F(E)$ such that $B_l=l\cap\overline{F(E)}\neq 0$. For a given support line l let b be a point in B_l. Then there exists an α such that the half plane defined by the set $\{e^{-i\alpha}[(1+z)/(1-z)]+b : z\in E\}$ contains $F(E)$. For this α and b, if F_1 is defined by $F_1(z)=e^{i\alpha}[F(z)-b]$ for z in E we have that $\mathrm{Re}\,F_1(z)>0$ for z in E. Thus we can apply the Lemma 1 to F_1 to obtain

$$\mathrm{Re}\left\{\frac{\phi*gF_1}{\phi*g}(z)\right\}=\mathrm{Re}\left\{e^{i\alpha}\frac{\phi*gF}{\phi*g}(z)-b\right\}>0,\quad z\in E.$$

Therefore, for each z in E we have that $[(\phi*gF)/(\phi*g)](z)$ lies in the appropriate half-plane for each support line l of $F(E)$. Hence, it follows that

$$[\phi*gF)/(\phi*g)](E)$$

lies in the convex hull of $F(E)$ as claimed.

Robinson's 1/2 conjecture is valid when X is replaced by K, S^* or C simply because $r_K[h_1]=1/2$. We shall now prove that X can also be replaced by S_p the class of spirallike functions. However, the result does not follow directly from the convexity of h_1 up to 1/2 because, unlike K, S^*, and C, S_p is not preserved under convolution with convex functions. We shall, however, still be able to obtain the result using convolution techniques by going directly to Lemma 2. We shall need the following lemma.

LEMMA 3. *Let f be in S and $F(z)=1+a_1z+\cdots$ be regular in E. Then, the image of $|z|<1/2$ under $(h_1*fF)/(h_1*f)$ is a subset of the convex hull of $F(E)$.*

Proof. This result follows from Lemma 2 upon showing that for all α and σ, $(|\alpha|=|\sigma|=1)$,

$$h_1(z)*\{[f(z)][(1+\alpha\sigma z)/(1-\sigma z)]\}=H(z)\neq 0 \quad \text{for } 0<|z|<1/2.$$

From the definition of h_1 we see that

$$2H(z)=\left(\frac{1+\alpha\sigma z}{1-\sigma z}\right)f(z)\left[1+\frac{zf'(z)}{f(z)}+\frac{(1+\alpha)\sigma z}{(1-\sigma z)(1+\alpha\sigma z)}\right].$$

Since f is in S, if we put $\zeta=\sigma z$, it is sufficient to show that for all $\alpha(|\alpha|=1)$

$$1+\frac{zf'(z)}{f(z)}+\frac{(1+\alpha)\zeta}{(1-\zeta)(1+\alpha\zeta)}\neq 0$$

for $|z|, |\zeta|<1/2$. We note that since f is in S we have that

$$|\log\{[zf'(z)]/f(z)\}|\leq\log[(1+r)/(1-r)] \quad \text{for } |z|\leq r.$$

Let $r=1/2$. We then have that

$$|\log\{[zf'(z)]/f(z)\}|\leq\log 3 \quad \text{for } |z|\leq 1/2.$$

Now, we claim that $|1+e^w|\geq 4/3$ if $|w|\leqslant\log 3$. To verify this, it clearly suffices to consider $w=-\rho e^{i\theta}$ for θ in $(-\pi,\pi]$ and $\rho>0$, and note that

$$|1+e^w|^2=1+e^{-2\rho\cos\theta}+2[\cos(\rho\sin\theta)]e^{-\rho\cos\theta}$$

It is clear that the replacement of θ by $-\theta$ does not change the above expression and thus it suffices to consider $0\leq\theta\leq\pi$. Letting $h(\rho,\theta)=|1+e^w|^2$, we see that

$$\frac{\partial h}{\partial\theta}=2\rho e^{-2\rho\cos\theta}[\sin\theta+\sin(\theta-\rho\sin\theta)e^{\rho\cos\theta}].$$

It now follows that $\partial h/\partial\theta\geq 0$ and therefore

$$|1+e^w|^2\geq h(\rho,0)=|1+e^{-\rho}|^2\geq(4/3)^2$$

and the claim follows. We now have that $|1+zf'(z)/f(z)|>4/3$ for $|z|<1/2$. Lemma 3 will follow upon showing that for all α, $(|\alpha|=1)$,

$$(2.3)\qquad \frac{(1+\alpha)\zeta}{(1-\zeta)(1+\alpha\zeta)}<\frac{4}{3}$$

provided $|\zeta|<1/2$. Inequality (2.3) will follow from our next lemma which will be used later in this paper.

LEMMA 4. *For* $|\alpha|=1$, *let* $f_\alpha(z)=[(1+\alpha)z]/[(1-z)(1+\alpha z)]$. *If* $|z|\leqslant r<1$, *then* $|f_\alpha(z)|\leqslant 2r/(1-r^2)$.

Proof. Write $\alpha=e^{2i\varphi}$, $-\pi/2<\varphi\leqslant\pi/2$ and $z=re^{i\theta}$. If we put $t=\theta+\varphi$ we see that

$$f_\alpha(z)=\frac{r(e^{-i\varphi}+e^{i\phi})e^{it}}{1+re^{it}(e^{i\varphi}-e^{-i\varphi})-r^2e^{2it}}$$
$$=\frac{2r\cos\varphi}{(1-r^2)\cos t+i[2r\sin\varphi-(1+r^2)\sin t]}$$

Now,

$$\begin{aligned}&|(1-r^2)\cos t+i[2r\sin\varphi-(1+r^2)\sin t]|^2\\ &=(1-r^2)^2+4r^2\sin^2 t-4r(1+r^2)\sin\varphi\sin t+4r^2\sin^2\varphi\\ &=4r^2\left[\sin t-\frac{(1+r^2)}{2r}\sin\varphi\right]^2+(1-r^2)^2\cos^2\varphi\\ &\geq(1-r^2)^2\cos^2\varphi\end{aligned}$$

It now follows that $|f_\alpha(z)|\leqslant 2r/(1-r^2)$.

With these results we can now prove that Robinson's conjecture is valid when S is replaced by S_p.

THEOREM 2.1. *The minimum radius of spirallikeness over all functions of the form* $1/2\ [f(z)+zf'(z)]$ *for* $f(z)\in S_p$ *is* $1/2$. *That is,* $r_{S_p}[\Gamma_1(S_p)]=1/2$.

Proof. Since f is in S_p there exists a real γ such that $H(z)=e^{i\gamma}zf'(z)/f(z)$ has positive real part in E. To show that $\Gamma_1 f=h_1*f$ is spirallike in $|z|<1/2$ we define H_1 by

$$H_1(z)=[e^{i\gamma}h_1*zf'(z)]/[h_1*f(z)]=[h_1*f(z)H(z)]/[h_1*f(z)].$$

Then Lemma 3 assures that $H_1[|z|<1/2]$ is contained in the convex hull of $H(E)$. The result follows by noting that $k(z)=z(1-z)^{-2}$ is spirallike for $\gamma=0$, and the radius of spirallikeness of $\Gamma_1[k]$ is $1/2$.

By using Lemma 2, we give another proof of Livingston's result given in [14, p. 356].

THEOREM 2.2. *Let* K_1 *be the class of functions* f *in* S *for which* Re $f'(z)>0$ *for* $z\in E$. *Then* $r_{K_1}[\Gamma_1(K_1)]=(\sqrt{5}-1)/2=r_0$

Proof. Since

$$(h_1*f)'(z)=[h_1(z)*zf'(z)]/z=[h_1(z)*zf'(z)]/[h_1(z)*z]$$

we need only show that Re $\{[h_1(z)*zf'(z)]/[h_1(z)*z]\}>0$ for $|z|<r_0$. By Lemma 2 it suffices to show that

$$H(z)=h_1(z)*z(1+\alpha\sigma z)/(1-\sigma z)\neq 0 \text{ for } 0<|z|<r_0,\ |\alpha|=|\sigma|=1.$$

However, from the definition of h_1 we have that

$$2H(z)=\frac{1+\alpha\sigma z}{1-\sigma z}2z+\frac{(1+\alpha)\sigma z^2}{(1-\sigma z)^2}=z\frac{1+\alpha\sigma z}{1-\sigma z}\left[2+\frac{(1+\alpha)\sigma z}{(1-\sigma z)(1+\alpha\sigma z)}\right]$$

By using Lemma 4, we obtain

$$\left|2+\frac{(1+\alpha)\sigma z}{(1-\sigma z)(1+\alpha\sigma z)}\right|\geq 2-\left|\frac{(1+\alpha)\sigma z}{(1-\sigma z)(1+\alpha\sigma z)}\right|\geq 2-2r/(1-r^2)>0,$$

$$0\leq r<r_0$$

The result then follows by considering the function

$$f_1(z)=-2\log(1-z)-z,$$

which shows that the result is sharp.

Let T be Rogosinski's class of typically real functions on E, noting that functions in T need not be univalent. Let C_1 be Robertson's class of functions in T that have their images convex in the direction of the imaginary axis. (see [19]). Recall Fejer's observation that h is in C_I if and only if zh' is in T. We include a new proof of Robertson's result in [20] showing that T is invariant under convolution with functions in C_I. We then give a corollary showing its application to Robinson's 1/2 conjecture. Let P be the class of functions p analytic in E which have positive real part and are normalized by $p(0)=1$.

THEOREM 2.3. *If h is in C_I and f is in T then $h*f$ is in T.*

Proof. It is a standard result that f is in T if and only if $f(z)=z(1-z^2)^{-1}p(z)$, where p is in P and has real coefficients. Also, for any function g in T, there exists a nondecreasing function μ_g on $[0,\pi]$ with $\mu_g(\pi)-\mu_g(0)=\pi$ and such that

$$g(z)=\frac{1}{\pi}\int_0^\pi\frac{z}{1-2z\cos t+z^2}d\mu_g(t).$$

For each t in $[0,\pi]$ the function g_t defined by

$$zg_t'(z)=z(1-2z\cos t+z^2)^{-1},\quad g(0)=0,$$

is convex because zg_t' is starlike. Thus, for t and s in $[0,\pi]$,

$$g(z,s,t)=zg_s'(z)*g_t(z)$$

is starlike and has real coefficients so that $g(z,t,s)$ is in T. Hence $g(z,s,t)=z(1-z^2)^{-1}p(z,s,t)$ where p is in P. By using these facts and the properties of convolution, we obtain for h in C_I that

$$(h*f)(z)=\frac{1}{\pi}\int_0^\pi h*\frac{z}{1-2z\cos t+z^2}d\mu_f(t)$$

$$=\frac{1}{\pi}\int_0^{\pi} zh'(z)*\int_0^{\pi}\frac{d\zeta}{1-2\zeta\cos t+\zeta^2}\,d\mu_f(t)$$

$$=\frac{1}{\pi}\int_0^{\pi}\frac{1}{\pi}\left[\int_0^{\pi}\frac{z}{1-2z\cos s+z^2}\,d\mu_{zh'}(s)\right]*g_t(z)\,d\mu_f(t)$$

$$=\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} zg_s'(z)*g_t(z)\,d\mu_{zh'}(s)\,d\mu_f(t)$$

$$=\frac{z}{1-z^2}\,\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} P(z,s,t)\,d\mu_{zh'}(s)\,d\mu_f(t)=\frac{z}{1-z^2}P_1(z)$$

where p_1 has real coefficients and is in P from the properties of $\mu_{zh'}$ and μ_f. Thus, from the characterization of T the theorem is proved.

COROLLARY. *If f is in S and has real coefficients then h_1*f is typically real for $|z|<1/2$.*

Proof. This follows from Theorem 2.3 since $r_K[h_1]=r_{C_I}[h_1]=1/2$ and any function in S with real coefficients is in T.

3. Generalization of Bernardi's and Burdick's results

We now give a generalization of Bernardi's results in [4]. Throughout Bernardi's paper he considered c a positive integer. We shall consider c a complex number with $c\neq -1$. In each case when c is considered a positive integer we obtain Bernardi's results. We define h_c by

$$h_c(z)=\sum_{n=1}^{\infty}\frac{n+c}{1+c}z^n=\frac{z-[c/(1+c)]z^2}{(1-z)^2}.$$

For f in A let the operator $\Gamma_c: A\longrightarrow A$ be defined by $\Gamma_c(f)=h_c*f$.

THEOREM 3.1. (i) *If $Re\,\{c\}>0$ then $r_C[\Gamma_c(K)]=1$.*

(ii) *If*

$$Re\,\{c\}>\frac{4r-1-r^2}{1-r^2} \tag{3.1}$$

and f is in K, S^* or C then h_c*f is convex, starlike, or close-to-convex, respectively, for $|z|<r$. If c is real and greater than -1 then

$$r_K[\Gamma_c(K)]=r_{S*}[\Gamma_c(S^*)]=r_C[\Gamma_c(C)]=r_0 \tag{3.2}$$

where $r_0=\{2-(3+c^2)^{1/2}\}/(1-c)$ for $c\neq 1$ and for $c=1$, $r_0=1/2$.

(iii) *If*

$$|c+1|>2r/(1-r^2) \tag{3.3}$$

and f is in K_1, *then Re* $\{(h_c*f)'(z)\}>0$ *for* $|z|<r$. *If c is real and greater than* -1, *then* $r_{K_1}[\Gamma_c(K_1)]=r_1$ *where* $r_1=[-1+(2+2c+c^2)^{1/2}]/(1+c)$.

Proof. Part (i) follows by the fact that h_c is in C if and only if $|c/_{(1+c)}-1/2|\leq 1/2$ which is equivalent to $\mathrm{Re}\{c\}>0$.

The first part of (ii) will follow by showing that whenever inequality (3.1) holds then $r\leq r_K[h_c]$. Consider, for $k(z)=z/(1-z)^2$, that

$$(3.4)\qquad 1+\frac{zh_c''(z)}{h_c'(z)}=\frac{h_c*k*k}{h_c*k}(z)=\frac{h_c*k[(k*k)/k]}{h_c*k}(z).$$

Since $(k*k)/k(z)=(1+z)/(1-z)$, Lemma 2 assures that the term on the right hand side of inequality (3.1) has positive real part whenever

$$(3.5)\qquad h_c*[k(1+\alpha\sigma z)/(1-\sigma z)]\neq 0 \qquad \text{for all } \alpha,\sigma,\ |\alpha|=|\sigma|=1$$

and $0<|z|<r$. Thus we need only show that this r is determined by the condition (3.1). From the definition of h_c and the comparison of their corresponding Taylor series we have that

$$(3.6)\qquad h_c*k\frac{1+\alpha\sigma z}{1-\sigma z}=\frac{1}{1+c}k\frac{1+\alpha\sigma z}{1-\alpha z}\left[c+\frac{zk'}{k}+\frac{(1+\alpha)\sigma z}{(1-\sigma z)(1+\alpha\sigma z)}\right]$$

Thus, it suffices to show that the bracketed term in (3.6) is nonzero for $|z|<r$ determined by (3.1). From the definition of $k(z)$ and Lemma 4 we have that

$$c+\frac{zk'(z)}{k(z)}+\frac{(1+\alpha)\sigma z}{(1-\sigma z)(1+\alpha\sigma z)}\neq 0$$

whenever Re $\{c+(1+z)/(1-z)\}>2r/(1-r^2)$. This holds when

$$(3.7)\qquad \mathrm{Re}\{c\}>\frac{2r}{1-r^2}-\frac{1-r}{1+r}=\frac{4r-1-r^2}{1-r^2}$$

as claimed. For real $c>-1$, (3.7) is equivalent to $c+1-4r+(1-c)r^2>0$, which holds whenever $0\leqslant r<r_0$. In order to complete the verification of (3.2) we consider the cases of sharpness. For the convex case we convolute $h_c(z)$ with $z/(1-z)$ to obtain

$$1+\frac{zh_c''(z)}{h_c'(z)}=\frac{(1+c)+4z+(1-c)z^2}{(1-z)[(1+c)+(1-c)z]}=J(z).$$

It easily follows that $J(-r_0)=0$. For the starlike and close-to-convex case we convolute h_c with $k(z)$ to obtain

$$\frac{z(h_c*k)'}{h_c*k}=1+\frac{zh_c''(z)}{h_c'(z)}=J(z).$$

So that again $J(-r_0)=0$. Since for $c=1$, $h_c*f=k_1*f=\Gamma_1 f$, the case $c=1$ follows from our previous results.

We prove (iii) by noting as before that

$$(h_c*f)'(z)=\frac{h_c(z)*zf'(z)}{h_c(z)*z}.$$

Since f is in K_1, Lemma 2 assures that Re $\{(h_c*f)'\}>0$ whenever

(3.8) $\qquad h_c*[z(1+\alpha\sigma z)/(1-\sigma z)]\neq 0, \quad |\alpha|=|\sigma|=1$

and $0<|z|<r$. We need only show that this r is determined by condition (3.3). We have

$$h_c*z\frac{1+\alpha\sigma z}{1-\sigma z}=\frac{1+\alpha\sigma z}{1-\sigma z}\left[1+c+\frac{(1+\alpha)\sigma z}{(1-\sigma z)(1+\alpha\sigma z)}\right]\frac{z}{1+c}$$

Thus applying Lemma 4, we have that inequality (3.8) holds whenever inequality (3.3) holds as claimed. For $c>-1$, inequality (3.3) is equivalent to

$$|z|<r_1=[-1+(2+2c+c^2)^{1/2}]/(1+c)$$

That $r_1=r_{K1}[\Gamma_c(K_1)]$ follows by convoluting $h_c(z)$ with $f_1(z)=-z-2\log(1-z)$ to obtain

$$(h_c*f_1)'(z)=[(1+c)+2z-(1+c)z^2]/(1-z)^2=g(z)$$

where $g(-r_1)=0$.

Another type of problem to which we can apply convolution techniques is the following: Let $G(z)=-f(-z)/f(z)$ and $F(z)=f'(-z)/f'(z)$. In [8] Burdick, Keogh, and Merkes determined the smallest α and β such that Re $\{G(z)\}^{\alpha}>0$ and Re $\{F(z)\}^{\beta}>0$, z in E, for f in K, S^* and C. Noting that

$$G(z)=[f(z)*z/(1+z)]/[f(z)*z/(1-z)]$$

and that $F(z)=[k(z)*f(z)*z/(1+z)]/[k(z)*f(z)*z/(1-z)]$ we obtain the following generalizations of their work. Given $\beta, 0\leqslant\beta<1$, let

$$S^*(\beta)=\{f\in S: \mathrm{Re}[zf'(z)/f(z)]>\beta, z\in E\}.$$

LLMMA 5. *If f is in $S^*(\beta)$ then*

(3.9) $\qquad Re\{[f(z)*z/(1+z)]/f(z)\}^{1/2(1-\beta)}>0, \quad z\in E.$

The result is sharp.

Proof. This lemma follows readily from the Herglotz representation for functions in $S^*(\beta)$. Since f is in $S^*(\beta)$ there exists a probability measure μ such that

$$\log[f(z)/z]=2(1-\beta)\int_{-\pi}^{\pi}\log(1-ze^{it})d\mu(t),$$

Thus we obtain

$$\frac{-f(-z)}{f(z)}=\exp\left[2(1-\beta)\int_{-\pi}^{\pi}\log\frac{1-ze^{it}}{1+ze^{it}}d\mu(t)\right],$$

and the result follows. The sharpness follows by considering

$$f(z)=k_\beta(z)-z(1-z)^{2(\beta-1)},\quad \text{where}\quad k_\beta(z)=\frac{z}{(1-e^{i\beta}z)^2}.$$

THEOREM 3.2. *If f is in C then*

$$Re\left\{\left[\frac{k_\beta(z)*f(z)*z/(1+z)}{k_\beta(z)*f(z)}\right]^{1/2(2-\beta)}\right\}> \;,\quad z\in E. \tag{3.10}$$

The result is sharp.

Proof. Let $K_\beta(z)=\int_0^z(k_\beta(\zeta)/\zeta)d\zeta$. Since f is in C, $zf'(z)=g(z)p(z)$ for $g(z)$ starlike and $\operatorname{Re}\{p(z)\}>0$, z in E. Let p_i designate a function with Re $\{p_i(z)\}>0$, z in E, for $i=1, 2, 3,$ and 4. Using these notations and Lemma 2, we have

$$\begin{aligned}
&\frac{k_\beta(z)*f(z)*\frac{z}{1+z}}{k_\beta(z)*f(z)}=\frac{K_\beta(z)*zf'(z)*\frac{z}{1+z}}{K_\beta(z)*zf'(z)}\\
&=\frac{K_\beta(z)*g(z)p_1(z)*\frac{z}{1+z}}{K_\beta(z)*g(z)p_1(z)}=\frac{[K_\beta(z)*g(z)]p_2(z)*\frac{z}{1+z}}{[K_\beta(z)*g(z)]p_2(z)}\\
&=\frac{\left\{[K_\beta(z)*g(z)]*\frac{z}{1+z}\right\}p_3(z)}{[K_\beta(z)*g(z)]p_2(z)}\\
&=\frac{\left\{\left[k_\beta(z)*\int_0^z\frac{g(\zeta)}{\zeta}d\zeta\right]*\frac{z}{1+z}\right\}p_3(z)}{\left[k_\beta(z)*\int_0^z\frac{g(\zeta)}{\zeta}d\zeta\right]p_2(z)}\\
&=\left\{\frac{\left[k_\beta(z)*\int_0^z\frac{g(\zeta)}{\zeta}d\zeta\right]*\frac{z}{1+z}}{k_\beta(z)*\int_0^z\frac{g(\zeta)}{\zeta}d\zeta}\right\}p_3(z)p_2^{-1}(z).
\end{aligned}\tag{3.11}$$

It is also easy to prove that if h is in K then $h*k_\beta$ is in $S^*(\beta)$. So from the convexity of $\int_0^z [g(\zeta)/\zeta]d\zeta$ and Lemma 5, we have that (3.11) equals

$$p_4^{2(1-\beta)}(z)p_3(z)p_1^{-1}(z).$$

Therefore, inequality (3.10) follows. Sharpness is proved by considering the function $f(z)=h_c(z)=\{z-[c/(1+c)]z^2\}/(1-z)^2$. A straightforward calculation gives that

$$\begin{aligned}&|\arg\{[k_\beta(z)*h_c(z)*z(1+z)^{-1}]/[k_\beta(z)*h_c(z)]\}|\\ (3.12)\qquad &=\left|\arg\left\{\left(\frac{1-z}{1+z}\right)^{3-2\beta}\left[\frac{1+[1-2\beta/(1+c)]z}{1-[1-2\beta/(1+c)]z}\right]\right\}\right|.\end{aligned}$$

If we let $1-2\beta/(1+c)=Re^{i\varphi}$ and $z=e^{i\theta}$, then, for $\theta=-\varphi+\pi/2$, (3.12) becomes

$$(3.13)\qquad |(3-2\beta)\pi/2+\text{Arc }\sin[2R/(1+R^2)]|.$$

Since R approaches 1 as $|c|$ approaches ∞ we have that (3.13) approaches $(2-\beta)\pi$ and the result follows.

4. Characterizations for the classes $\mathcal{R}(\alpha,\beta)$ and $Q(\alpha,\beta)$

We now give two generalizaions of $\mathcal{R}_\alpha$, the class of functiotns prestarlike of order α. Let $\alpha<1$ and $\beta<1$.

A function $f\in A$ normalized by $f(0)=f'(0)-1=0$, is said to be in $\mathcal{R}(\alpha,\beta)$ if $f*s_\alpha\in\mathcal{S}_\beta$. Note that $\mathcal{R}(\alpha,\alpha)\equiv\mathcal{R}_\alpha$.

A function $f\in A$ normalized by $f(0)=f'(0)-1=0$, is said to be in $Q(\alpha,\beta)$ if $f*g\in\mathcal{S}_\beta$ for all $g\in\mathcal{S}_\alpha$. For $\alpha\le\beta$, the class $Q(\alpha,\beta)$ was studied by Ruscheweyh and Singh [25]. Suffridge [33] showed that $Q(\alpha,\alpha)\equiv\mathcal{R}(\alpha,\alpha)$

The following lemma which is due to Ruscheweyh is useful in our generalizations of $\mathcal{R}_\alpha$

LEMMA 6. [28]. *The function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ in A is prestarlike of order α $(\alpha<1)$ if and only if*

$$(4.1)\qquad f(z)*\frac{z-xz^2}{(1-z)^{3-2\alpha}}\neq 0\qquad (0<|z|<1,\ \ |x|=1).$$

We begin with a discussion of the relationship of $Q(\alpha,\ \beta)$ to $\mathcal{R}(\alpha,\beta)$

for $\alpha \neq \beta$. Since $s_\alpha \in \mathcal{S}_\alpha$ it is clear that

$$Q(\alpha, \beta) \subset \mathcal{R}(\alpha, \beta) \tag{4.2}$$

We determine when this containment is proper with the following

THEOREM 4.1. *We have $Q(\alpha, \beta) = \mathcal{R}(\alpha, \beta)$ if and only if $\beta \geq \alpha$.*

Proof. For $\beta \geq \alpha$, let f be in $\mathcal{R}(\alpha, \beta)$ so that $f * s_\alpha * s_\beta^{-1} \in \mathcal{R}_\beta$. From (1.2), $g \in \mathcal{S}_\alpha$ implies $g * s_\alpha^{-1} \in \mathcal{R}_\alpha \subset \mathcal{R}_\beta$. It follows from (1.3) that

$$(f * s_\alpha * s_\beta^{-1}) * (g * s_\alpha^{-1}) = f * g * s_\beta^{-1} \in \mathcal{R}_\beta,$$

or, equivalently, that $f * g \in \mathcal{S}_\beta$ for all $g \in \mathcal{S}_\alpha$. Therefore, $f \in Q(\alpha, \beta)$. Thus, $\mathcal{R}(\alpha, \beta) \subset Q(\alpha, \beta)$ which combined with (4.2) proves that $Q(\alpha, \beta) \equiv \mathcal{R}(\alpha, \beta)$.

For $\beta < \alpha$, we show that $s_\beta * s_\alpha^{-1}$ in $\mathcal{R}(\alpha, \beta)$ is not in $Q(\alpha, \beta)$. It is known [30] that $z + b_n z^n \in \mathcal{S}_\gamma$ if and only if

$$|b_n| \leq \frac{1-\gamma}{n-\gamma}. \tag{4.3}$$

Hence,

$$s_\beta * s_\alpha^{-1} * \left(z + \frac{1-\alpha}{2-\alpha} z^2\right) = z + \frac{1-\beta}{2-\alpha} z^2 \notin \mathcal{S}_\beta \quad \text{for } \beta < \alpha.$$

For $0 \leqslant \alpha < 1$, an application of Theorem 4.1 solves the problem of determining the order of starlikeness of a function known to be convex of order α. This problem was solved by MacGregor [15] using the princple of subordination. We obtain the solution simply as a

COROLLARY. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}_\alpha$ $(0 \leq \alpha < 1)$, then f is starlike of order*

$$\beta(\alpha) = \begin{cases} \dfrac{4^\alpha(1-2\alpha)}{4-2^{2\alpha+1}}, & \alpha \neq 1/2, \\ \dfrac{1}{\log 4}, & \alpha = 1/2. \end{cases}$$

Proof. It is known that $f \in \mathcal{K}_\alpha$ implies f is starlike of order at least α. We want to find $\beta = \beta(\alpha) \geq \alpha$ such that $zf' = f * s_0 \in \mathcal{S}_\alpha$ implies $f \in \mathcal{S}_\beta$, i.e., $s_0^{-1} \in Q(\alpha, \beta)$. By Theorem 4.1, $s_0^{-1} \in Q(\alpha, \beta)$ if and only if

$$s_0^{-1} * s_\alpha = \int_0^z (1-\zeta)^{2\alpha-2} d\zeta \in \mathcal{S}_\beta.$$

Thus, $\beta(\alpha)$ is the order of starlikeness of

$$f_{\alpha}(z)=\begin{cases}-\dfrac{1}{2\alpha-1}((1-z)^{2\alpha-1}-1), & \alpha\neq 1/2,\\ \log\left(\dfrac{1}{1-z}\right), & \alpha=1/2,\end{cases}$$

which is the value of $\beta(\alpha)$ recently given by Wilken and Feng [35].

THEOREM 4. 2. *A necessary and sufficient condition for f to be in $\mathcal{R}(\alpha, \beta)$ is that*

$$G(\alpha, z)=\frac{f(z)*(s_{\alpha}(z)/(1-z))}{f(z)*s_{\alpha}(z)}$$

satisfies

$$\text{(4.4)} \qquad \operatorname{Re} G(\alpha, z)\geq\frac{1+\beta-2\alpha}{2(1-\alpha)} \qquad (z\in E).$$

Proof. If $f\in\mathcal{R}(\alpha, \beta)$, then $g=s_{\alpha}*f\in\mathcal{S}_{\beta}$. From the identity

$$\frac{s_{\alpha}(z)}{1-z}=s_{\alpha}(z)*\left\{\frac{1-2\alpha}{2-2\alpha}\,\frac{z}{1-z}+\frac{1}{2-2\alpha}\,\frac{z}{(1-z)^{2}}\right\},$$

it follows that

$$f(z)*\frac{s_{\alpha}(z)}{1-z}=\frac{1-2\alpha}{2-2\alpha}g(z)+\frac{1}{2-2\alpha}zg'(z).$$

Therefore,

$$G(\alpha, z)=\frac{1-2\alpha}{2-2\alpha}+\frac{1}{2-2\alpha}\,\frac{zg'(z)}{g(z)}$$

and (4. 4) follows. Conversely, (4. 4) implies that $g\in\mathcal{S}_{\beta}$, and hence $f\in\mathcal{R}(\alpha, \beta)$.

Given a family of functions $V\subset A$, the dual of V, denoted by V^{*}, is defined by

$$V^{*}=\{f\in A: f*g\neq 0 \text{ for } \forall g\in V,\ z\in E\}.$$

The double dual of V is defined by $(V^{*})^{*}$ and is denoted by V^{**}. Dual spaces in this context were introduced by Ruscheweyh [23], who demonstrated their importance in solving various extremal problems. In [23], some families were given whose duals were various well known subclasses of S.

COROLLARY. *Ths function f is in $\mathcal{R}(\alpha, \beta)$ if and only if*

$$(4.5)\qquad f*\frac{z+\dfrac{(1-\alpha)x+\beta-\alpha}{1-\beta}z^2}{(1-z)^{3-2\alpha}}\neq 0 \quad (|x|=1,\ 0<|z|<1).$$

Proof. From Theorem 4.2, if $f\in\mathcal{R}(\alpha,\beta)$, then $f*s_\alpha\neq 0$ $(0<|z|<1)$ and

$$(4.6)\qquad \frac{f(z)*\dfrac{s_\alpha(z)}{1-z}}{f(z)*s_\alpha(z)}\neq 1-\left(\frac{1-\beta}{1-\alpha}\right)\left(\frac{1}{1+x}\right),\quad |x|=1.$$

A simplification of (4.6) leads to (4.5). Conversely (4.5) implies (4.6), which yields (4.4) by continuity.

In order to get characterization results for $Q(\alpha,\beta)$ we need the following definitions and remarks that were originally given in [29].

Suppose that $k(z)$ is analytic and nonzero in E and that λ is real. We say $k\in\Pi_\lambda$ if

$$\operatorname{Re}\frac{zk'(z)}{k(z)}\begin{cases}<\lambda/2 & \text{for } \lambda>0\\ >\lambda/2 & \text{for } \lambda<0\\ \equiv 0 & \text{for } \lambda=0\end{cases}$$

For $f\in A$, we say $f\in K(\alpha,\beta)$ $(\alpha\geq 0,\ \beta\geq 0)$, if f can be written in the form $f(z)=k(z)H(z)$ where $k\in\Pi_{\alpha-\beta}$ and $H\in A$ is nonzero and satisfies

$$|\arg H(z)|\leq\frac{1}{2}\pi\min(\alpha,\beta)\quad (z\in E).$$

REMARKS [29]. (1) For $\lambda<0$, $k\in\Pi_\lambda$ if and only if $zk\in\mathcal{S}_{1+(\lambda/2)}$.

(2) Finite products of the form $k(z)=c\Pi_{j=1}^{n}(1+x_kz)^{\lambda k}$ where $|x_k|=1$, $c\neq 0$, and $\sum_{k=1}^{n}\lambda_k=\lambda$ (all λ_k having the same sign) are dense in Π_λ.

(3) For $f\in K(\alpha,\beta)$, $f\neq 0$.

(4) If $g\in\Pi_\alpha$ and $h\in\Pi_\beta$ then $f=g/h\in K(\alpha,\beta)$.

In [29] it is shown that $K(\alpha,\beta)$ lies in the second dual of the class of functions

$$(4.7)\qquad \frac{(1+xz)^m(1+uz)^\gamma}{(1-yz)^\beta}\quad (|x|=|y|=|u|=1 \text{ or } x=u=-y \text{ and } |x|<1)$$

where $m=[\alpha]$ is the largest integer not exceeding α and $m+\gamma=\alpha$. In particular, for $m=1$ and $\beta=3-2\alpha$ in (4.7), we have that $K(1,3\text{-}2\alpha)$ lies in $\{(1+xz)/(1+yz)^{3-2\alpha}\}^{**}$. From Lemma 6, it follows that

(4.8) $$K(1,\ 3\text{-}2\alpha)\subset\left\{\frac{f}{z}\ :\ f\in\mathcal{R}_\alpha\right\}^*$$

The dual of $\mathcal{R}_\alpha$ provides us with nice necessary and sufficient conditions for a function to be in $Q(\alpha,\beta)$ when $\beta<\alpha$.

Theorem 4.3. *A necessary and sufficient condition for f to be in $Q(\alpha,\beta)$ is that*

$$f*s_\alpha*s_\beta^{-1}*\frac{z(1+xz)}{(1-z)^{3-2\beta}}\in\mathcal{R}_\alpha^*\quad(|x|=1).$$

Proof. We know that

$$f*s_\alpha*s_\beta^{-1}*\frac{z(1+xz)}{(1-z)^{3-2\beta}}\in\mathcal{R}_\alpha^*$$

if and only if

(4.9) $$\varphi*f*s_\alpha*s_\beta^{-1}*\frac{z(1+xz)}{(1-z)^{3-2\beta}}\neq 0\qquad(\text{for all }\varphi\in\mathcal{D}_\alpha).$$

By Lemma 6, (4.9) is equivalent to requiring that

(4.10) $$f*s_\alpha*s_\beta^{-1}*\varphi\in\mathcal{R}_\beta\qquad\text{for all }\varphi\in\mathcal{R}_\alpha.$$

But (4.10) is a necessary and sufficient condition for $f*s_\beta^{-1}*g$ to be in $\mathcal{R}_\beta$ for all $g\in\mathcal{S}_\alpha$. Hence $f*g\in\mathcal{S}_\beta$ for all $g\in\mathcal{S}_\alpha$, and consequently $f\in Q(\alpha,\beta)$. From (4.8) and Theorem 4.3, we have the following

Corollary. *A sufficient condition for f to be in $Q(\alpha,\beta)$ is that*

(4.11) $$\frac{f(z)}{z}*\frac{s_\alpha(z)}{z}*\frac{s_\beta^{-1}(z)}{z}*\frac{1+xz}{(1-z)^{3-2\beta}}\in K(1,3-2\alpha)$$

for $|x|=1$.

The usefulness of (4.11) in providing information about $Q(\alpha,\beta)$ for particular choices of α and β is nicely illustrated for the case $\alpha=1/2$ and $\beta=0$. With these choices (4.11) becomes

(4.12) $$\frac{f(z)}{z}*\frac{1}{z}\log\left(\frac{1}{1-z}\right)*\frac{1+xz}{(1-z)^3}\in K(1,2)$$

for each x, $|x|=1$. A simple calculation shows that (4.12) is equivalent to

(4.13) $$\frac{f(z)}{z}*\frac{1+\frac{1}{2}(x-1)z}{(1-z)^2}\in K(1,2)\quad(|x|=1).$$

Since

$$\frac{1+\frac{1}{2}(x-1)z}{(1-z)^2}=\frac{x+1}{2}\frac{1}{(1-z)^2}-\frac{x-1}{2}\frac{1}{1-z},$$

it follows from (4.13) that

$$\frac{1}{2}(x+1)f'(z)-\frac{1}{2}(x-1)\frac{f(z)}{z}\in K(1,2), \quad |x|=1.$$

Equivalently, for all real t.

$$\frac{f(z)}{z}\left(\frac{zf'(z)}{f(z)}+it\right)\in K(1,2) \tag{4.14}$$

is a sufficient condition for f to be in $Q(1/2,0)$.

Sheil-Small [29] obtained the following properties of $K(\alpha,\beta)$, which will prove useful in applications of (4.14).

LEMMA 7. [29]. *(a) If $\alpha'\leqslant\alpha$, $\beta'\leqslant\beta$, then $K(\alpha',\beta')\subset K(\alpha,\beta)$.*

(b) If $f\in K(\alpha,\beta), g\in K(\alpha',\beta')$ then $fg\in K(\alpha+\alpha', \beta+\beta')$.

(c) The function f is in $K(\alpha,\alpha)$ if and only if $|arg\ (e^{i\mu}f)|\leq\alpha\pi/2$ for some μ real.

THEOREM 4.4. *If there exists λ, $0\leqslant\lambda\leqslant1$, such that*

$$\left|arg\frac{zf'(z)}{f(z)}\right|\leqslant\frac{\pi\lambda}{2} \tag{i}$$

and

$$\frac{f(z)}{z}\in K\left(\frac{1-\lambda}{2},\ \frac{3-\lambda}{2}\right) \tag{ii}$$

then $f\in Q(1/2,0)$.

Proof. It follows from (i) that, for $t>0$, $-\lambda\pi/2\leqslant\arg(zf'/f+it)\leqslant\pi/2$ while, for $t<0$, $\pi/2\leqslant\arg(zf'/f+it)\leqslant\lambda\pi/2$. In either case, there exists a real μ such that

$$\left|\arg\left[e^{i\mu}\left(\frac{zf'(z)}{f(z)}+it\right)\right]\right|\leqslant\frac{1}{2}\left|\frac{\pi}{2}-\left(\frac{-\lambda\pi}{2}\right)\right|=\frac{1+\lambda}{2}\frac{\pi}{2}. \tag{4.15}$$

From Lemma 7, part (c), (4.15) is equivalent to requiring that for all real t,

$$\frac{zf'(z)}{f(z)}+it\in K\left(\frac{1+\lambda}{2},\ \frac{1+\lambda}{2}\right) \quad (z\in E)$$

which, combined with condition (ii) and Lemma 7, part (b), proves (4.14).

The theorem provides us with an indication of some of the geometry that can be sufficient to describe a function in $Q(1/2, 0)$. For example, we deduce the following

COROLLARY. *If*

$$\left|\arg z\frac{f'(z)}{f(z)}\right|\leqslant\frac{\pi\lambda}{2}\quad and\quad \mathrm{Re}\frac{zf'(z)}{f(z)}\geq\frac{\lambda+1}{4}\quad (0\leqslant\lambda\leqslant 1,\ \ z\in E),$$

then $f\in Q(1/2, 0)$.

Proof. It is clear that $\Pi_r = K(0, -r)$ for $\gamma<0$. For $g=f/z$, we have

$$\mathrm{Re}\,\frac{zg'(z)}{g(z)}=\mathrm{Re}\frac{zf'(z)}{f(z)}-1\geq\frac{\lambda+1}{4}-1=\frac{\lambda-3}{4}=-\left(\frac{3-\lambda}{4}\right).$$

It follows that $f/z\in K(0,\ (3-\lambda)/2)$ which is contained in $K((1-\lambda)/2,\ (3-\lambda)/2)$ by Lemma 7, part (a). The result is now a consequence of the theorem.

We conclude this section with a coefficient criterion for a function to be in $Q(\alpha, \beta)$, which is a consequence of the corollary to Theorem 4.3.

THEOREM 4.5. *If* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ *satisfies*

$$\sum_{n=2}^{\infty}\frac{n-\beta}{1-\beta}C(\alpha, n)\,|a_n|\leq 1,$$

then $f\in Q(\alpha, \beta)$.

Proof. A direct computation yields that

$$\begin{aligned}&\frac{f(z)}{z}*\frac{s_\alpha(z)}{z}\ \frac{*s_\beta^{-1}(z)}{z}*\frac{1+xz}{(1-z)^{3-2\beta}}\\ &=\sum_{1}^{\infty}\frac{n+1-2\beta+(n-1)x}{2(1-\beta)}C(\alpha, n)a_nz^{n-1}\\ &=1+w(z, x)\end{aligned}$$

where $|w(z, x)|\leqslant|z|<1$. Thus, $1+w(z, x)\in K(1, 1)\subset K(1, 3-2\alpha)$ and (4.11) is satisfied.

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