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A GENERALIZATION OF PRIME IDEALS IN SEMIGROUPS

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In [3], Murata and his coauthors defined f-prime ideals in rings and obtained analogous results of Van der Walt [4]. In this paper, f-prime ideals in semigroups are defined and obtained results similar to those in [3]. One found that the f-radical of an ideal A of a semigroup defined by the author is the intersection of all f-prime ideals containing A. Under the left regularity assumption, the radical of an ideal A turns out to be the f-radical of A. Moreover, the properties of primary ideals in semigroups [1] such as the uniqueness of decomposition theorem by Laske-Noether could be extended for f-primary ideals.

1. f-prime ideals and the f-radical of an ideal

Throughout, S will denote a semigroup and F will denote the set of all functions f from S into the set of all ideals in S such that, for each s in S,

(1) $s \in f(s)$,

(2) $x \in f(s)$ implies $f(x) \subset f(s)$,

(3) $x \in f(s) \cup A$ implies $f(x) \subset f(s) \cup A$ for each ideal A of S.

It is clear that the function f defined by $f(s)=(s)$, the principal ideal generated by s, is in F. For a fixed ideal B of S, the function defined by $f(s)=(s) \cup B$ is also in F.

Definition. A subset Q of S is called a p-system iff $(a)(b) \cap Q \neq \phi$ for any a, b in Q. Q is said to be an sp-system iff $(a)^2 \cap Q \neq \phi$ for each a in Q.

It is evident that every subsemigroup of S is a p-system and every p-system is an sp-system. Let $S=\{a, b, c, d\}$ be the semigroup with the

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following multiplication table:

	a	b	c	d
a	a	a	a	a
b	a	b	a	a
c	a	a	c	a
d	a	a	a	d

As is easily seen, $\{a, b\}$ is a p-system and $\{b, c, d\}$ is an sp-system which is not a p-system.

DEFINITION. For $f \in F$, a subset Q of S is called an f-system [sf-system] iff it contains a p-system [sp-system] Q^* such that $Q^* \cap f(q) \neq \phi$ for each q in Q. In each case, Q^* will be called a kernel of Q.

A proper ideal P in S is called f-prime [f-semiprime] iff its complement P^c is an f-system [sf-system].

It is clear that every f-prime ideal is f-semiprime.

A proper ideal P of S is completely prime iff $xy \in P$ for some x, y in S implies $x \in P$ or $y \in P$. A proper ideal P of S is prime if $XY \subset P$ where X and Y are ideals of S implies $X \subset P$ or $Y \subset P$.

In a commutative semigroup with identity, every prime ideal is completely prime. Every completely prime ideal in S is f-prime, but the converse is not true.

EXAMPLE (1) Let N be the semigroup of positive integers with the usual product. Consider a function f from N into the set of all ideals in N which is defined by $f(n) = 3N \cup nN$. It is clear that f is contained in F. Let $P = 4N$ and $Q^* = 3N - 6N$. Then $Q^* \subset P^c$ and for any q_1, q_2 in Q^*, $(q_1)(q_2) \cap Q^* \neq \phi$ which proves that Q^* is a p-system. Since $f(q) \cap Q^* \neq \phi$ for any $q \in P^c$, the ideal P is f-prime. But P is not prime. In this case, every prime ideal is f-prime.

(2) Let $T = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ x + y \leq 1\}$ be a triangle semigroup under $(x, y)(x', y') = (xx', xy' + y)$. Consider a function f from T into the set of all ideals in T defined by $f((x, y)) = ((x, y)) \cup ((\frac{1}{2}, 0))$. Then $f \in F$. Since $(1, 0)$ is a unit and $(x, y)T \subset T(x, y)$, $((x, y)) =$

$T(x, y)$. Let $P=T\left(\frac{1}{4}, \frac{3}{4}\right)$. Take $Q^*=\{(x, 0)\mid 0<x\leq 1\} \subseteq P^c$, it is clearly that Q^* is a p-system. Since $f(a)\cap Q^*\neq\phi$ for any $a\in P^c$, P is f-prime but not prme. For, $\left(\left(\frac{1}{2}, \frac{1}{2}\right)\right)=T\left(\frac{1}{2}, \frac{1}{2}\right)\not\subset P$ and $\left(\left(\frac{1}{3}, \frac{1}{2}\right)\right)$ $=T\left(\frac{1}{3}, \frac{1}{2}\right)\not\subset P$. Since $(x, y)\,T\subset T(x, y)$, $T\left(\frac{1}{2}, \frac{1}{2}\right)T\left(\frac{1}{3}, \frac{1}{2}\right)\subset TT$ $\left(\frac{1}{2}, \frac{1}{2}\right)\left(\frac{1}{3}, \frac{1}{2}\right)\subset T\left(\frac{1}{6}, \frac{3}{4}\right)\subset P$.

Proposition 1.1. *For any f-prime [f-semiprime] ideal P of S, $f(a_1)\ f(a_2)\subset P$ implies $a_1\in P$ or $a_2\in P$ [$f(a)^2\subset P$ implies $a\in P$].*

Proof. Suppose $a_i\in P^c(i=1, 2)$. Since P^c is an f-system, there exists a p-system $Q^*\subset P^c$ such that $f(a_i)\cap Q^*\neq\phi$ $(i=1, 2)$. Let $x_1\in f(a_1)\cap Q^*$ and $x_2\in f(a_2)\cap Q^*$. Then $(x_1)(x_2)\cap Q^*\neq\phi$ and hence $f(x_1)f(x_2)\cap Q^*\neq\phi$ which is a contradiction. The proof of the other half could be done similarly.

It is clear that the union of prime ideals in S is prime. However, the (finite) union of f-prime ideals in S need not be f-prime. In Example (1), let $P_1=3N$ and $P_2=4N\cup 6N$. Then $f(2)f(2)\subset P_1\cup P_2$ $=3N\cup 4N$ and $2\notin P_1\cup P_2$. Then by Proposition 1.1, $P_1\cup P_2$ is not f-prime.

Let A be any ideal of S. Then the ideal $\bigcup_{a\in A} f(a)$ is denoted by $f(A)$. Clearly $A\subset f(A)$ and $f(A)\subset f(B)$ if $A\subset B$. Moreover, $f(a)=f((a))$ since $x\in(a)\subset f(a)$ implies $\bigcup_{x\in(a)} f(x)\subset f(a)$. In general, $f(A)\neq A$. But if $f(a)=(a)$, then $f(A)=A$.

Proposition 1.2. *Let P be an f-prime [f-semiprime] ideal of S. T. A. E.*

(i) *$f(a)\,f(b)\subset P$ implies $a\in P$ or $b\in P$ [$f(a)^2\subset P$ implies $a\in P$]*

(ii) *$f(A)f(B)\subset P$ implies $f(A)\subset P$ or $f(B)\subset P$, for any ideals A, B of S [$f(A)^2\subset P$ implies $f(A)\subset P$].*

Proof. Obviously (ii) implies (i). Let a, b in P^c, then $f(a)\cap P^c\neq\phi$ and $f(b)\cap P^c\neq\phi$. Since $f(a)=f((a))$, $f((a))\cap P^c\neq\phi$ and $f((b))\cap P^c\neq\phi$. Thus $f((a))f((b))\cap P^c\neq\phi$ implies $f(a)f(b)\cap P^c\neq\phi$. The proof of the other half is similar.

Definition. A subset A of S is called *semiprime* iff for $a\in S$, $a^2\in A$

implies $a\in A$.

COROLLARY 1.3. *If $f(a)=(a)$ for each a in S, then prime and f-prime are synonyms. Moreover, under the same condition, semiprime and f-semiprime are synonyms whenever S is commutative.*

DEFINITION. Let A be an ideal of S. Then $r_f(A)=\{x|Q\cap A\neq\phi$ for each f-system Q containing $x\}$, $r_{sf}(A)=\{x|Q\cap A\neq\phi$ for each sf-system Q containing $x\}$ will be called the *f-radical and sf-radical* of A respectively.

THEOREM 1.4. *Let A be an ideal of S. Then $r_f(A)$ [$r_{sf}(A)$] is the intersection of all f-prime [f-semiprime] ideals of S.*

Proof. Let C be the intersection of all f-prime ideals containing A. It is clear that $r_f(A)\subset C$. Conversely, if $x\notin r_f(A)$, then there exists an *f-system* Q such that $x\in Q$ and $Q\cap A=\phi$. Let P be the union of all ideals B such that $A\subset B$ and $B\cap Q=\phi$ and let Q^* be a kernel of Q. Then $Q^*\subset P^c$. For any element a in P^c, $A\subset f(a)\cup P$ and P is maximal with respect to the properties $A\subset P$ and $P\cap Q=\phi$. Since $P\subsetneq f(a)\cup P$, $(f(a)\cup P)\cap Q\neq\phi$. Thus $f(a)\cap Q\neq\phi$ and there exists q in Q such that $q\in f(a)$. By a property of f, $f(q)\subset f(a)$. Since Q is an f-system, $f(q)\cap Q^*\neq\phi$. It follows that $f(a)\cap Q^*\neq\phi$ and P^c is an f-system with the kernel Q^*. Hence P is f-prime and $x\notin P$, i.e., $C\subset r_f(A)$.

For any ideal A of S, we denote

$$\bar{A}=\{x\in S|f(x)^n\subset A \text{ for some positive integer } n\}$$
$$A'=\{x\in S|x^n\in A \text{ for some positive integer } n\}.$$

Let $x\in\bar{A}$. Then $f(x)^n\subset A\subset r_f(A)$ for some n. Hence $x\in r_f(A)$ by Proposition 1.1. Thus $\bar{A}\subset r_f(A)$. Let $x\in S$ and $x^n\notin A$ for all n. Then $\{x, x^2, \ldots, x^n, \ldots\}$ is an f-system of S and $\{x, x^2, \ldots\}\cap A=\phi$. Hence $x\notin r_f(A)$ and $r_f(A)\subset A'$. Therefore, $\bar{A}\subset r_f(A)\subset A'$.

THEOREM 1.5. *Let A be an ideal of a left regular semigroup S. Then $r_f(A)=A'$ for any $f\in F$.*

Proof. Suppose $x\notin r_f(A)$. It is well known that S is left regular iff every left ideal of S is semiprime [5]. Hence A is semiprime. It follows that for each positive integer n, $x^n\in A$ implies $x\in A$. Therefore $x\notin A$ implies $x^n\notin A$ for each n. Hence $x\notin A'$.

Let Q^* be a p-system such that $Q^* \cap A = \phi$. Let C be the collection of all p-systems which contain Q^* and do not meet A. Since $Q^* \in C$, C is nonempty. It is clear that the union of a chain in C is in C, and hence C has a maximal element M^*. Let $M = \{x \in S \mid f(x) \cap M^* \neq \phi\} \cap A^c$. Then M is an f-system with the kernel M^* and $M \cap A = \phi$. As is seen in the proof of Theorem 1.4, there exists an f-prime ideal P such that $A \subset P$ and $P \cap M = \phi$. Since P^c is an f-system with the kernel M^*, $P^c = M$.

Definition. An f-prime ideal P is called a minimal f-prime ideal belonging to an ideal A iff P contains A and there exists a kernel Q^* for the f-system P^c such that Q^* is a maximal p-system which does not meet A.

It is clear that any f-prime ideal P containing A contains a minimal f-prime ideal belonging to A and the f-radical of an ideal A coincides with the intersection of all minimal f-prime ideals belonging to A.

In general, an aribitrary intersection of f-prime ideals of S may not be f-prime. However, an arbitrary intersection of f-semiprime ideals of S is f-semiprime. It follows that an arbitrary intersection of f-prime ideals of S is f-semiprime, and an ideal A in S is f-semiprime iff $r_f(A) = A$.

2. f-primary ideals

Definition. An element a is (*right*) *f-related* to an ideal A of S iff for each $b \in f(a)$, there exists an element $c \notin A$ such that $cb \in A$. An ideal B is (*right*) *f-related* to an ideal A of S iff every element of B is f-related to A.

Lemma 2.1. *Let A be an ideal of S and let K be the set of all elements of S which are not f-related to A. Then K is an f-system.*

Proof. Let q be an element of K. Then there exists b in $f(q)$ such that $cb \notin A$ for every element $c \notin A$. Let K^* be the set of all such b. Then K^* is a p-system and $f(q) \cap K^* \neq \phi$. Hence K is an f-system with the kernel K^*

In Example (1), let $A = 4N$ and $f(a) = aN \cup 3N$ for any $a \in S$. Then $3 \in f(a)$ and $3(4n+i) \notin A$ for $i = 1, 2, 3$. It follows that for any $c \notin A$, $3c \notin A$. Hence A is not f-related to A. However, each element of a proper ideal A is f-related to A if f is defined to be $f(a) = (a)$ for each

a in S.

For the rest of this section, we assume that
(α) Every ideal A of S is f-related to A

PROPOSITION 2.2. *The f-radical $r_f(A)$ of an ideal of S is f-related to A.*

Proof. Let K be the set of all elements of S which are not f-related to A. Suppose $x \in r_f(A)$ and x is not f-related to A. Then by Lemma 2.1, K is an f-system containing x. It follows that $K \cap A \neq \phi$, which contradicts the assumption (α).

Let K be the set of all elements of S which are not f-related to A. Then K is an f-system and $K \cap A = \phi$ by Lemma 2.1 and the assumption (α). Let P be the union of all ideals which are f-related to A and do not meet K. As the proof of Theorem 1.4, P becomes f-prime. This unique maximal ideal P will be called the maximal f-prime ideal belonging to A. By the assumption (α), P contains A. Since an element x is f-related to an ideal A iff $f(x)$ is f-related to A, every element f-related to A is contained in P.

For ideals A and B of S and $x \in S$, we adopt the notation $A : x = \{y \in S \mid f(y)f(x) \subset A\}$ and $A : B = \cap \{A : x \mid x \in B\}$

PROPOSITION 2.3. *Let A be an ideal of S and $b \in S$. If $A : b \neq \phi$, then $A : b$ is an ideal containing A.*

Proof. Let $x \in A : b$ and $s \in S$. Then $x \in f(x)$ and $xs \in f(x)$. It follows that $f(xs) \subset f(x)$ and $f(xs)f(b) \subset f(x)f(b) \subset A$. Thus $xs \in A : b$. Similary, $sx \in A : b$. Let $a \in A$ and $x \in A : b$. Then $xa \in A : b \cap A$, and $f(xa)f(b) \subset A$. For any $a' \in A$, $f(a') \subset f(xa) \cup A$ since $a' \in f(xa) \cup A$. Then $f(a')\ f(a) \subset (f(xa) \cup A)f(b) = f(xa)f(b) \cup A\, f(b) \subset A$, and hence $a' \in A : b$.

Let P be the maximal f-prime ideal belonging to an ideal A of S and let

$$A_p = \begin{cases} \bigcup_{s \notin P} (A : s) & \text{if } P \neq S \\ A & \text{if } P = S. \end{cases}$$

If $f(a) = (a)$, for any a of S, then $A_p \neq \phi$ since $A \subset A : s$ for any

$s \notin A$. In Example (1), let $A=4N$ and $P=2N$. Then for any $s \in S$, $9N \subset f(x)f(s)$. It follows that $A : s = \{x \in S \mid f(x)f(s) \subset 4N\} = \phi$, and hence $A_p = \phi$ whenever $P \neq S$.

For the rest of this section, we will also assume that

(β) For any ideals A and B with $B \not\subset r_f(A)$, $A : B \neq \phi$.

Proposition 2.4. *Let P be the maximal f-prime ideal belonging to an ideal A of S. Then $A = A_p$.*

Proof. By the assumption (β), $A_p \neq \phi$. For any element x in A_p, there exists $s \in P^c$ such that $f(x)f(s) \subset A$. Since s is not f-related to A, there exists $s' \in f(s)$ such that $cs' \in A$ implies $c \in A$. Then $xs' \in A$, and hence $x \in A$. Therefore $A = A_p$.

Definition. Let K be an f-system in S. A kernel K^* of K is said to be dense in K iff $K^* \cap A \neq \phi$ for any ideal A in S with $K \cap A \neq \phi$.

If $f(a) = (a)$ for any a in S, then every kernel K^* of an f-system K is dense in K. However, in Example (1), since $P = 4N$ is f-prime, P^c is an f-system with the kernel $K^* = 3N - 6N$. Then $K^* \cap 6N = \phi$ while $P^c \cap 6N \neq \phi$, and hence K^* is not dense in P^c.

Definition. An ideal A of S is (right) f-primary iff $f(a)f(b) \subset A$ implies $a \in A$ or $b \in r_f(A)$.

Every f-prime ideal must be f-primary by Proposition 1.1.

Proposition 2.5. *Let A and B be ideals of S. Then*

(1) *$A \subset B$ implies $r_f(A) \subset r_f(B)$*

(2) *$r_f(r_f(A)) = r_f(A)$*

(3) *$r_f(AB) = r_f(A \cap B) = r_f(A) \cap r_f(B)$ if every f-system in S has a dense kernel.*

Proof. Clearly (1) and (2) hold. Now $r_f(AB) \subset r_f(A \cap B) \subset r_f(A) \cap r_f(B)$ by (1). Let $x \in r_f(A) \cap r_f(B)$ and let K be any f-system containing x. Then $K \cap A \neq \phi$ and $K \cap B \neq \phi$. Since K has the dense kernel K^*, $K^* \cap A \neq \phi$ and $K^* \cap B \neq \phi$. Let $a \in K^* \cap A$, $b \in K^* \cap B$. Then $(a)(b) \cap K^* \neq \phi$. Since $(a)(b) \subset AB$, $AB \cap K^* \neq \phi$ and hence $AB \cap K \neq \phi$, which means $x \in r_f(AB)$.

Corollary 2.6. *Assume that every f-system in S has a dense kernel.*

Let Q and T be f-primary ideals such that $r_f(Q)=r_f(T)$. Then $Q\cap T$ is an f-primary ideal and $r_f(Q\cap T)=r_f(Q)=r_f(T)$.

PROPOSITION 2.7. *An ideal A is f-primary iff $A:B=A$ for every ideal $B\not\subset r_f(A)$.*

Proof. Suppose A is f-primary and B is an ideal such that $B\not\subset r_f(A)$. By the assumption (β), $A:B\neq\phi$ implies $A\subset A:B$. Let $b\in B$ and $b\notin r_f(A)$. For each element $x\in A:B$, $x\in A$ since A is f-primary. Hence $A:B\subset A$ implies $A:B=A$. Conversely, suppose $f(a)f(b)\subset A$ and $b\notin r_f(A)$. Then $f(b)\not\subset r_f(A)$. Hence $A:f(b)=A$ implies $f(a)f(b')\subset f(a)f(b)\subset A$, for every $b'\in f(b)$. Therefore $a\in\cap\{A:b'\,|\,b'\in f(a)\}=A:f(b)=A$.

DEFINITION. If an ideal A can be written as $A=A_1\cap A_2\cap\ldots\cap A_n$, where A_i is an f-primary ideal for each i, it is called an f-primary decomposition of A. Every A_i is called an f-primary component of A.

A decomposition is called irredundant iff $\bigcap_{i\neq j} A_j\not\subset A_i$ for each i.

An irredundant f-primary decomposition is said to be reduced iff $r_f(A_i)\neq r_f(A_j)$ $(i\neq j)$.

If an ideal A of S has an f-primary decomposition and if every f-system in S has a dense kernel, then A has a reduced f-primary decomposition by Corollary 2.6.

In the rest of this section, we assume the following:

(γ) $A:A=S$ for any f-primary ideal A.

In Example (1), let $A=4N$. Since $9N\subset f(x)f(a)\not\subset A$ for $a\in A$ and $x\in S$, $A:A=\phi$. Thus the assumption (γ) is essential. However, (γ) holds if $f(a)=(a)$ for every a in S.

THEOREM 2.8. *Let $A=A_1\cap A_2\cap\ldots\cap A_n=A'_1\cap A'_2\cap\ldots\cap A'_m$ be two reduced f-primary decompositions of A. Then $n=m$ and it is possible to renumber the f-primary components in such a way that $r_f(A_i)=r_f(A'_i)$ for $1\leq i\leq n=m$.*

Proof. Using Proposition 2.5, Proposition 2.7 and Corollary 2.6, the proof follows as in Theorem 3.7 of [3].

3. f-primary semigroups

PROPOSITION 3.1. *Let A be an ideal of a semigroup S with identity* 1.

If $r_f(A)=S-H(1)$, *then* A *is* f*-primary. Where* $H(1)$ *is the maximal subgroup containing* 1.

Proof. Let $f(x)f(y)\subset A$ and $x\notin A$. Suppose $y\notin r_f(A)$. Then $f(y)\not\subset r_f(A)=S-H(1)$, and hence $f(y)=S$. Then $f(x)f(y)=f(x)S=f(x)\subset A$. and $x\in A$ which is a contradiction. Thus A is f-primary.

PROPOSITION 3.2. *Let* S *be a semigroup with identity* 1 *and let every* f*-system in* S *has a dense kernel. Then for any* $n\in N$, M^n *is* f*-primary, where* $M=S-H(1)$.

Proof. By Proposition 2.5 (3), $r_f(M^n)=r_f(M)\cap\ldots\cap r_f(M)=M\cap\ldots\cap M$. Hence M^n is f-primary by Proposition 3.1.

DEFINITION. A semigroup S is called f-primary iff every ideal of S is f-primary.

THEOREM 3.3. *Let* S *be a semigroup with identity* 1. *If* S *has no* f*-prime ideal except* $S-H(1)$, *then* S *is an* f*-primary semigroup. The converse is not true as in shown in* [2].

Proof. Let A be a proper (nonzero) ideal. Then $r_f(A)=S-H(1)$. By Proposition 3.1, A is f-primary.

THEOREM 3.4. *Let* S *be a left regular semigroup. If the set of all* f*-prime ideals of* S *is linearly ordered, then* S *is* f*-primary.*

Proof. Let A be an ideal of S and let $f(x)f(y)\subset A$. If $x\notin A$, $x^n\notin A$ for each positive integer n by the left regularity of S. Then $x\notin r_f(A)$ by Theorem 1.5. Since f-prime ideals are linearly ordered, $r_f(A)$ is f-prime. Now, since $f(x)f(y)\subset r_f(A)$, $y\in r_f(A)$ by Proposition 1.1.

References

1. A. Anjaneyulu, *primary ideals in semigroups*, Semigroup Forum Vol. **20** (1980) 129-144.
2. M. Satyanarayaha, *Commutative primary semigroups*, Czechosbvak Math. Vol. **22** (1972) 509-516.
3. Murata K., Kurata Y., and Marubayashi H., *A generalization of prime ideals in rings*, Osaka J. Math. **6**(1969) 291-301.
4. A.P.J. Van der Walt, *Contributions to ideal theory in general rings*, Proc.

Kon. Ned. Akad. Wetensch. **A67**(1964) 68-77.
5. Clifford and Preston, *Algebraic theory of semigroups I. II,* American Mathematical Society.

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