# ON GENERALIZED KKM-THEOREMS

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Let $E$ be a Hausdorff topological vector space. We shall denote by $2^E$ the set of all nonempty subsets of $E$. For $A \in 2^E$, $\overline{A}$ denotes its closure, and $co(A)$ denotes the convex hull of $A$. Let $X$ be a nonempty subset of $E$. A map $T : X \rightarrow 2^E$ is called a KKM-map if $co\{x_1, ..., x_n\} \subset \bigcup_{i=1}^{n} T(x_i)$ for each finite subset $\{x_1, ..., x_n\}$ of $X$. Note that the KKM-condition tells us only about the union of $T(x)'s$. We do not need any topological restrictions on $T$. It is clear that a KKM-map $T$ must be a multimap. Studying various branches of analysis, particularly convex analysis, we frequently meet some KKM-maps, which have a number of important applications (see [4, 6, 10, 14, 18, etc.]).

Recall that the classical KKM-theorem takes the following form:

THEOREM [12]. *Let $X$ be the set of vertices of a simplex in $E = \mathbf{R}^n$, and let $T : X \rightarrow 2^E$ be a compact valued KKM-map. Then $\bigcap_{x \in X} T(x) \neq \phi$.*

It is well-known that the KKM-theorem is equivalent to the Brouwer fixed point theorem and the celebrated Sperner lemma [18]. Those three results underpin many powerful results in broad areas in mathematical sciences. All are extremely important and, although seemingly different, are in a deep sense equivalent. For the details, see [18].

For the first time, the KKM-theorem was only used in fixed point theory [12]. Later examples of applications of the KKM-theorem were in dimension theory [13], mathematical economics [8], and minimax problem [16]. Since Fan's generalization of the KKM-theorem was appeared in 1961, many more applications have been obtained by a number of authors in studying invariant subspaces of linear operators [9], fixed point theory [3], variational inequalities [2], quasi-variational

inequalities [15], mathematical economics and game theory [1], and so on. Also many extensions of Fan's generalization have been obtained by a number of authors for various purposes, e. g., Brézis-Nirenberg-Stampacchia [2], Dugundji and Granas [4], Fan [6, 7] and Lassonde [14].

The purpose of this paper is to generalize and unify numerous generalizations of the KKM-theorem. Before proceeding further, we quote the infinite dimensional version of the KKM-theorem due to Fan, which is the first basic theorem of this paper.

THEOREM 0 ([5]). *Let $X$ be a subset of a Hausdorff topological vector space $E$, and let $T : X \to 2^E$ be a closed valued KKM-map. If $T(x_0)$ is compact for at least one $x_0 \in X$, then $\bigcap_{x \in X} T(x) \neq \phi$.*

There are two possibilities to generalize Theorem 0. On the one hand, the closedness assumption on $T(x)$ can be relaxed, e. g., finitely closedness [4], relatively closedness [6] and compactly closedness [14]. On the other hand, the compactness assumption on $T(x_0)$ can also be relaxed. In fact, the requirement that $T(x_0)$ be compact for some $x_0 \in X$ is not always met in practice [2]. The compactness assumption only used to assure that any family of closed sets, having the finite intersection property, admits the whole intersection property. Therefore, we only need some weaker compactness assumptions to assure the whole intersection property, e. g., $c$-compactness [14].

Now we introduce more general closedness conditions, which are relative versions of Dugundji-Granas [4] and Lassonde [14].

DEFINITION. Let $Y$ be a nonempty subset of a topological space $E$. A set $X \subset Y$ is called a *finitely relatively closed* subset of $Y$ if the intersection of $X$ with any finite dimensional subspace $F$ of $E$ is a relatively closed subset of $Y \cap F$. A set $X \subset Y$ is called a *compactly relatively closed* subset of $Y$ if the intersection of $X$ with any compact subset $K$ of $E$ is a relatively closed subset of $Y \cap K$.

Note that every finitely closed subset of $E$ is necessarily finitely relatively closed, and every compactly closed subset of $E$ is also compactly relatively closed. Moreover, every relatively closed subset is also finitely relatively closed and compactly relatively closed. Note that if $Y$ is closed, then the relative versions of Definition are equivalent to the corresponding ones in [4, 14].

Let $Y$ be a nonempty subset of a Hausdorff topological vector space $E$. A set $X \subset Y$ is called a *precompact* subset of $Y$ if $X$ is contained in some compact subset of $Y$. In other words, the closure of $X$ is a compact subset of $Y$. Every compact set is clearly precompact, and every precompact set is not necessarily compact.

We are now ready to give generalizations of the KKM-theorem. Each of the following theorems contains Theorem 0 and the KKM-theorem as special cases.

LEMMA. *Let $Y$ be a convex substet of a Hausdorff topological vector space $E$, and $\phi \neq X \subset Y$. Let $T : X \to 2^E$ be a KKM-map such that each $T(x)$ is a relatively closed subset of $Y$. Furthermore, assume that there exists a nonempty subset $X_0 \subset X$, contained in some precompact convex subset $Y_0$ of $Y$, such that $\overline{\bigcap_{x \subset X_0} T(x)}$ is a compact subset of $Y$. Then $\bigcap_{x \in X} T(x) \neq \phi$.*

*Proof.* For any finite subset $\{x_1, ..., x_n\}$ of $X$, let $X_1 = X_0 \cup \{x_1, ..., x_n\}$. Since $Y_0$ is a precompact convex subset of $Y$, the convex hull of $\overline{Y_0} \cup \{x_1, ..., x_n\}$ is also a compact convex subset of $Y$, and denote it by $K$. For each $y \in X_1$, let $G(y) = T(y) \cap K$. Since $T(y)$ is closed in $Y$, and $K$ is a compact subset of $Y$, each $G(y)$ is also compact. Furthermore, since $T$ is a KKM-map, we can easily show that $G$ is also a KKM-map. Therefore, by Theorem 0, we have $\bigcap_{y \subset X_1} G(y) \neq \phi$. Hence we have

$$\phi \neq \bigcap_{x \subset X_1} G(x) = \bigcap_{x \in X_1} T(x) \cap K$$
$$\subset \overline{\bigcap_{x \in X_0} T(x)} \cap T(x_1) \cap \cdots \cap T(x_n).$$

Let $C$ denote the compact set $\overline{\bigcap_{x \in X_0} T(x)}$. Then we have $\bigcap_{i=1}^{n} T(x_i) \cap C \neq \phi$ for every finite subset $\{x_1, ..., x_n\} \subset X$. Since each $T(x)$ is a relatively closed subset of $Y$ and $C$ is a compact subset of $Y$, each $T(x) \cap C$ is also a compact subset of $Y$. Since the family $\{T(x) \cap C \mid x \in X\}$ has the finite intersection property, we have

$$\bigcap_{x \subset X} T(x) \cap C = \bigcap_{x \in X} T(x) \neq \phi.$$

This completes the proof.

REMARK. In a recent paper [7], Fan shows the contrapositive of a weaker form of the above lemma by using the KKM-theorem with a

lemma.

THEOREM 1. *Let $Y$ be a convex subset of a Hausdorff topological vector space $E$, and $\phi \neq X \subset Y$. Let $T: X \rightarrow 2^E$ be a KKM-map such that each $T(x)$ is a finitely relatively closed subset of $Y$. Furthermore, assume the following*:

(1) *There exists a nonempty finite dimensional set $X_0 \subset X$, contained in some precompact convex subset of $Y$, such that $\overline{\bigcap_{x \in X_0} T(x)}$ is a compact subset of $Y$.*

(2) *For every line segment $L$ of $E$ we have*

$$\overline{(\bigcap_{x \in X \cap L} T(x))} \cap L = \bigcap_{x \in X \cap L} T(x) \cap L.$$

*Then $\bigcap_{x \in X} T(x) \neq \phi$.*

*Proof.* The conclusion holds in finite dimensional case by Lemma. Let $\{E_i | i \in I\}$ be the class of all finite dimensional subspaces of $E$, containing $X_0$ as subset, ordered by inclusion, i.e., $i \geq j$ if and only if $E_j \subset E_i$. Then, by the finite dimensional case, for each $i \in I$ there exists a point

$$y_i \in \bigcap_{x \in X \cap E_i} T(x) \cap E_i.$$

Let $\phi_i = \{y_j | j \geq i\}$ for each $i \in I$. Then the family $\{\phi_i | i \in I\}$ has the finite intersection property and $\phi_i \subset \bigcap_{x \in X_0} T(x)$ for each $i \in I$. Since $\overline{\bigcap_{x \in X_0} T(x)}$ is compact, $\bigcap_{i \in I} \overline{\phi_i} \neq \phi$, so there exists a point $y \in \bigcap_{i \in I} \overline{\phi_i}$. We show that $y$ is contained in each $T(x)$. For any $x \in X$, we consider the line segment $L$, which joins $x$ and $y$. Then $L$ is contained in some $F_i$ for sufficiently large $i \in I$. Hence, we have

$$y \in \overline{\phi_i} \cap L \subset \overline{(\bigcap_{z \in X \cap E_i} T(z))} \cap L$$
$$\subset (\overline{\bigcap_{z \in X \cap L} T(z)}) \cap L$$
$$= \bigcap_{z \in X \cap L} T(z) \cap L.$$

Therefore $y \in T(x)$, and consequently $y \in \bigcap_{x \in X} T(x)$. This completes the proof.

In case of $Y = E$ in Theorem 1, we obtain the following generalization of the Brézis–Nirenberg–Stampacchia lemma [2].

COROLLARY 2. *Let $X$ be a nonempty subset of a Hausdorff topological*

*vector space* $E$, *and let* $T : X \to 2^E$ *be a KKM-map such that each* $T(x)$
*is finitely closed. Furthermore, assume the following*:

(1) *There exists a nonempty finite dimensional set* $X_0 \subset X$, *contained
in some precompact convex subset of* $E$, *such that* $\overline{\bigcap_{x \in X_0} T(x)}$ *is compact.*

(2) *For every line segment* $L$ *of* $E$ *we have*

$$\left( \overline{\bigcap_{x \in X \cap L} T(x)} \right) \cap L = \bigcap_{x \subset X \cap L} T(x) \cap L.$$

Then $\bigcap_{x \subset X} T(x) \neq \phi$.

REMARKS. (i) In Theorem 1 and Corollary 2, the condition (2) can
be replaced by the following without affecting the conclusion.

(2′) *For every finite dimensional subspace* $E_i$ *of* $E$ *we have*

$$\left( \overline{\bigcap_{x \in X \cap E_i} T(x)} \right) \cap E_i = \bigcap_{x \subset X \cap E_i} T(x) \cap E_i.$$

(ii) Theorem 1 and Corollary 2 are generalizations of the Brézis-
Nirenberg–Stampacchia lemma [2]. In fact, the authors used the follo-
wing strong conditions instead of (1) and (2) (or (2′)):

(a) $\overline{T(x_0)}$ is compact for some $x_0 \in X$.

(b) For every convex subset $D$ of $E$,

$$\left( \overline{\bigcap_{x \subset X \cap D} T(x)} \right) \cap D = \bigcap_{x \in X \cap D} T(x) \cap D.$$

(iii) The Brézis-Nirenberg–Stampacchia lemma has often been used
in recent papers [10, 17, etc]. Hence we can generalize the results of
those papers by relaxing the hypotheses.

Now we prove the following intersection property by using the com-
pactly relatively closedness concept:

THEOREM 3. *Let* $Y$ *be a convex subset of a Hausdorff topological vector
space* $E$, *and* $\phi \neq X \subset Y$. *Let* $T : X \to 2^E$ *be a KKM-map such that each*
$T(x)$ *is a compactly relatively closed subset of* $Y$. *Furthermore, assume
that there exists a nonempty set* $X_0 \subset X$, *contained in some precompact
convex subset* $Y_0$ *of* $Y$ *such that* $\overline{\bigcap_{x \in X_0} T(x)}$ *is a compact subset of* $Y$.

*Then* $\bigcap_{x \in X} T(x) \neq \phi$.

*Proof.* Let $\{x_1, ..., x_n\}$ be a finite subset of $X$. Since $Y_0$ is a pre-
compact convex subset of $Y$, the convex hull $K$ of $\overline{Y_0} \cup \{x_1, ..., x_n\}$ is
a compact convex subset of $Y$. Now we define a multimap $G : X \cap K \to$
$2^E$ by $G(y) = T(y) \cap K$ for each $y \in X \cap K$. Since $T(y)$ is compactly

closed in $Y$, $G(y)$ is a compact subset of $Y$. Moreover, the multimap $G$ is clearly a KKM-map. Therefore, by Theorem 0, $\bigcap_{x \in X \cap K} G(x) \neq \phi$. Since

$$\phi \neq \bigcap_{x \in X \cap K} G(x) = \bigcap_{x \in X \cap K} T(x) \cap K$$
$$\subset (\overline{\bigcap_{x \in X_0} T(x)}) \cap T(x_1) \cap \cdots \cap T(x_n)$$

for every finite subset $\{x_1, \ldots, x_n\}$ of $X$, we have

$$\bigcap_{i=1}^{n} T(x_i) \cap (\overline{\bigcap_{x \in X_0} T(x)}) \neq \phi.$$

Since each $T(x)$ is a compactly relatively closed subset of $Y$ and $\overline{\bigcap_{x \in X_0} T(x)}$ is a compact subset of $Y$, we have

$$\bigcap_{x \in X} T(x) \cap (\overline{\bigcap_{x \in X_0} T(x)}) = \bigcap_{x \in X} T(x) \neq \phi.$$

This completes the proof.

REMARKS. (i) Theorem 3 is clearly a generalization of the previous lemma, so Theorem 3 includes generalizations of the KKM-theorem due to Fan [5, 6, 7].

(ii) In Theorem 3, the compactness of $\overline{\bigcap_{x \in X_0} T(x)}$ does not exclude the possibility that it is empty. However, the conclusion $\bigcap_{x \in X} T(x) \neq \phi$ of the theorem implies that $\overline{\bigcap_{x \in X_0} T(x)}$ is necessarily nonempty.

COROLLARY 4. Let $X$ be a nonempty subset of a Hausdorff topological vector space $E$, and let $T : X \to 2^E$ be a KKM-map such that each $T(x)$ is compactly closed. Assume that there exists a nonempty subset $X_0 \subset X$, contained in some precompact convex subset of $E$, such that $\overline{\bigcap_{x \in X_0} T(x)}$ is compact. Then $\bigcap_{x \in X} T(x) \neq \phi$.

COROLLARY 5. Let $X$ be a nonempty subset of a Hausdorff topological vector space $E$, and let $T : X \to 2^E$ be a KKM-map such that each $T(x)$ is compactly closed. If $\overline{T(x_0)}$ is compact for some $x \in X$, then $\bigcap_{x \in X} T(x) \neq \phi$.

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