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SEMI-INVARIANT SUBMANIFOLDS WITH PARALLEL NORMAL CURVATURE IN A COMPLEX SPACE FORM*

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0. Introduction

Recently Nakagawa, Umehara and one of the present authors [6] studied the submanifolds with harmonic curvature in a Riemannian manifold of constant curvature.

On the other hand Blair, Ludden and Yano [1] introduced the semi-invariant immersion. From this point of views, Yano and one of the present authors [10] investigated semi-invariant submanifolds of real codimension 3 in a complex Euclidean space.

The purpose of the present paper is to study semi-invariant submanifolds with parallel normal curvature in a complex space form and to characterize the submanifolds with harmonic curvature.

We use the systems of indices throughout this paper as follows;

$$A, B, C, \ldots = 1, \ldots, 2n+1, \ldots, 2n+4,$$
$$h, i, j, \ldots = 1, \qquad \ldots, 2n+1.$$

The summation convention will be used with respect to those systems of indices.

1. Submanifolds admitting almost contact metric structure

Let $(\overline{M}, G)$ be an almost Hermitian manifold of real dimension $2(n+2)$ equipped with an almost complex structure J and with an almost Hermitian metric tensor G. Let $\overline{M}$ be covered by a system of coordinate neighborhoods $\{\bar{U}\,; y^A\}$ and denote by G_{BA} the components of G and by $J_B{}^A$ those of J. Then we have

$$(1.1) \qquad J_A{}^B J_B{}^C = -\delta_A{}^C, \quad G_{CD} J_B{}^C J_A{}^D = G_{BA}.$$

Let M be a $(2n+1)$-dimensional Riemannian manifold covered by a

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system of coordinate neighborhoods $\{U\,;\,x^h\}$ and immersed isometrically in $\bar{M}$ by the immersion $\phi: M \longrightarrow \bar{M}$. When the argument is local, M need not be distinguished from $\phi(M)$. We represent the immersion ϕ locally by $y^A=y^A(x^h)$ and put $B_j{}^A=\partial_j y^A$, $(\partial_j=\partial/\partial x^j)$. Then $B_j=(B_j{}^A)$ are $(2n+1)$-linearly independent local tangent vectors of M and denote by C, D and E three mutually orthogonal unit normals to M. Then the induced Riemannian metric g_{ji} on the submanifold M is given by $g_{ji}=B_j{}^A B_i{}^C G_{AC}$ because the immersion ϕ is isometric. By denoting by ∇_j the operator of van der Waerden-Bortolotti covariant differentiation formed with g_{ji}, equations of Gauss and Weingarten for M are respectively obtained:

$$\nabla_j B_i{}^A = h_{ji}C^A + k_{ji}D^A + l_{ji}E^A, \tag{1.2}$$

$$\nabla_j C^A = -h_j{}^h B_h{}^A + l_j D^A + m_j E^A, \tag{1.3}$$

$$\nabla_j D^A = -k_j{}^h B_h{}^A - l_j C^A + n_j E^A, \tag{1.4}$$

$$\nabla_j E^A = -l_j{}^h B_h{}^A - m_j C^A - n_j D^A, \tag{1.5}$$

where h_{ji}, k_{ji} and l_{ji} are the second fundamental forms in the direction of C, D and E respectively and l_j, m_j and n_j the third fundamental tensors of M.

The transformations of $B_i{}^A$, C^A, D^A and E^A by the almost complex structure J are represented in each coordinate neighborhood as follows:

$$J_C{}^A B_i{}^C = f_i{}^h B_h{}^A + u_i C^A + v_i D^A + w_i E^A, \tag{1.6}$$

$$J_B{}^A C^B = -u^h B_h{}^A - \nu D^A + \mu E^A, \tag{1.7}$$

$$J_C{}^A D^C = -v^h B_h{}^A + \nu C^A - \lambda E^A, \tag{1.8}$$

$$J_C{}^A E^C = -w^h B_h{}^A - \mu C^A + \lambda D^A, \tag{1.9}$$

where we have put $f_{ji}=G(JB_j, B_i)$, $u_i=G(JB_i, C)$, $v_i=G(JB_i, D)$, $w_i=G(JB_i, E)$ and u^h, v^h and w^h being vector fields associated with u_i, v_i and w_i respectively, λ, μ, ν being functions in M. From these definitions we verify that f_{ji} is skew-symmetric and the functions λ, μ and ν are globally defined on M. By the properties of the almost Hermitian structure, it follows from (1.6)～(1.9) that

$$\left\{\begin{array}{l} f_i{}^t f_t{}^h = -\delta_i^h + u_i u^h + v_i v^h + w_i w^h, \\ f_t{}^h u^t = \nu v^h - \mu w^h, \quad f_t{}^h v^t = -\nu u^h + \lambda w^h, \\ f_t{}^h w^t = \mu u^h - \lambda v^h, \quad u_t u^t = 1-\mu^2-\nu^2, \\ v_t v^t = 1-\nu^2-\lambda^2, \quad w_t w^t = 1-\lambda^2-\mu^2, \\ u_t v^t = \lambda\mu, \quad u_t w^t = \lambda\nu, \quad v_t w^t = \mu\nu. \end{array}\right. \tag{1.10}$$

If we define a vector field p^h by

$$p^h = \lambda u^h + \mu v^h + \nu w^h, \tag{1.11}$$

it is easily, using (1.10), seen that

(1.12) $$f_t{}^h p^t = 0.$$

We put $N^A = \lambda C^A + \mu D^A + \nu E^A$. Then it is an intrinsically defined normal to M [10]. Thus (1.7)～(1.9) can be written as follows:

(1.13) $$J_C{}^A N^C = -p^h B_h{}^A.$$

Suppose that the functions λ, μ and ν satisfy $\lambda^2 + \mu^2 + \nu^2 = 1$. Then, by definition, we easily see that it is a global condition on M. Thus the equation (1.6) reduces to

(1.14) $$J_C{}^A B_i{}^C = f_i{}^h B_h{}^A + p_i N^A,$$

because C^A, D^A and E^A are mutually orthogonal unit normals to M, where $p_i = p^t g_{ti}$. We also see from (1.11) that p^h defines a unit vector field on M. By the properties of the almost Hermitian structure, it follows from (1.12)～(1.14) that (f, g, p) defines an almost contact metric structure.

Conversely, if the set (f, g, p) of the tensor field f of type (1, 1), the Riemannian metric tensor g and the vector field p given by (1.11) defines an almost contact metric structure. We can show, taking account of (1.10)～(1.11), that $\lambda^2 + \mu^2 + \nu^2 = 1$.

On the other hand, a submanifold M of an almost Hermitian manifold is called a *CR-submanifold* [11] if there is a differentiable distribution $T : p \longrightarrow T_p \subseteq M_p$ on M satisfying the following conditions, where M_p denotes the tangent space at each point p in M: (1) T is invariant, i.e., $JT_p = T_p$ for each p in M, (2) The complementary orthogonal distribution $T^\perp : p \longrightarrow T_p{}^\perp \subseteq M_p$ is totally real, i.e. $JT_p{}^\perp \subseteq M_p{}^\perp$ for each p in M, where $M_p{}^\perp$ denotes the normal space to M at $p \in M$. In particular, M is said to be *semi-invariant* provided that dim $T^\perp = 1$. In this case a unit normal in $JT^\perp$ is called the *distinguished normal* to the semi-invariant submanifold.

Thus, if M is a semi-invariant submanifold of codimension 3 in $\overline{M}$ with respect to the distinguished normal $N^A = \lambda C^A + \mu D^A + \nu E^A$, then the set (f, g, p) defines an almost contact metric structure [4], [10] and hence $\lambda^2 + \mu^2 + \nu^2 = 1$.

2. Semi-invariant submanifolds of a complex space from

In the sequel, the ambient Hermitian manifold $\overline{M}$ is assumed to be of constant holomorphic sectional curvature $4c$ and of real dimension $2(n+2)$, which is called a *complex space form* and denoted $\overline{M}(c)$.

Let M be a $(2n+1)$-dimensional semi-invariant submanifold of codimension 3 in $\overline{M}(c)$ and denote by N^A the distinguished normal to M. Then we have (1.13) and (1.14) because $\lambda^2+\mu^2+\nu^2=1$ is satisfied. We take $N^A=\lambda C^A+\mu D^A+\nu E^A$ as C^A. Then it follows that $\lambda=1$, $\mu=\nu=0$ and consequently $u^h=p^h$, $v_i=0$, $w_i=0$ because of (1.10) and (1.11).

Thus (1.6)$\sim$(1.9) reduces respectively to

$$J_C{}^A B_i{}^C = f_i{}^h B_h{}^A + p_i C^A, \tag{2.1}$$

$$J_B{}^A C^B = -p^h B_h{}^A, \quad J_C{}^A D^C = -E^A, \quad J_C{}^A E^C = D^A. \tag{2.2}$$

If we apply the operator ∇_j of the covariant differentiation to (2.1) and (2.2) and make use of the equations from (1.1) to (1.5), then we get respectively (cf. [4], [10])

$$\nabla_j f_i{}^k = -h_{ji} p^k + h_j{}^k p_i, \quad \nabla_j p_i = -h_{jt} f_i{}^t, \tag{2.3}$$

$$k_{ji} = -l_{jt} f_i{}^t - m_j p_i, \quad l_{ji} = k_{jt} f_i{}^t + l_j p_i. \tag{2.4}$$

Therefore, it is immediately from (2.4) that

$$k_{jt} p^t = -m_j, \quad l_{jt} p^t = l_j, \quad k = -m_t p^t, \quad l = l_t p^t, \tag{2.5}$$

where we have put $k=\sum_i k_{ii}$, $l=\sum_i l_{ii}$. Since (f, g, p) defines an almost contact metric structure, it is easily seen from (2.3)$\sim$(2.5) that

$$f_i{}^t l_t = k p_i + m_i, \quad kl + m_t l^t = 0, \tag{2.6}$$

$$k_{jt} l_i{}^t + k_{it} l_j{}^t = -(l_j m_i + l_i m_j), \tag{2.7}$$

$$l_{jt} l_i{}^t - k_{jt} k_i{}^t = l_j l_i - m_j m_i. \tag{2.8}$$

Since the ambient manifold is a complex space form $\overline{M}(c)$, its curvature tensor is given by

$$R_{DCBA} = c(G_{DA} G_{CB} - G_{CA} G_{DB} + J_{DA} J_{CB} - J_{CA} J_{CB} - 2 J_{DC} J_{BA}).$$

Thus it follows from (1.2)$\sim$(1.5), (2.1) and (2.2) that the equations of Gauss, Codazzi, and Ricci for M are respectively obtained:

$$\begin{aligned} R_{kjih} = {} & c(g_{kh} g_{ji} - g_{jh} g_{ki} + f_{kh} f_{ji} - f_{jh} f_{ki} - 2 f_{kj} f_{ih}) \\ & + h_{kh} h_{ji} - h_{jh} h_{ki} + k_{kh} k_{ji} - k_{jh} k_{ki} + l_{kh} l_{ji} - l_{jh} l_{ki}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} & \nabla_k h_{ji} - \nabla_j h_{ki} - l_k k_{ji} + l_j k_{ki} - m_k l_{ji} + m_j l_{ki} \\ & \quad = c(p_k f_{ji} - p_j f_{ki} - 2 p_i f_{kj}), \end{aligned} \tag{2.10}$$

$$\nabla_k k_{ji} - \nabla_j k_{ki} + l_k h_{ji} - l_j h_{ki} - n_k l_{ji} + n_j l_{ki} = 0, \tag{2.11}$$

$$\nabla_k l_{ji} - \nabla_j l_{ki} + m_k h_{ji} - m_j h_{ki} + n_k k_{ji} - n_j k_{ki} = 0, \tag{2.12}$$

$$\nabla_k l_j - \nabla_j k_k + h_k{}^t k_{jt} - h_j{}^t k_{kt} + m_k n_j - m_j n_k = 0, \tag{2.13}$$

$$\nabla_k m_j - \nabla_j m_k + h_k{}^t l_{jt} - h_j{}^t l_{kt} + n_k l_j - n_j l_k = 0, \tag{2.14}$$

$$\nabla_k n_j - \nabla_j n_k + k_k{}^t l_{jt} - k_j{}^t l_{kt} + l_k m_j - l_j m_k = 2c f_{kj}. \tag{2.15}$$

Now, we denote the normal component of $\nabla_j C$ by $\nabla_j{}^{\perp}C$. The normal vector field C is said to be *parallel* in the normal bundle if $\nabla_j{}^{\perp}C=0$, namely, l_j and m_j vanish identically. From now on we suppose that the normal vector C is parallel in the normal bundle. Then the equations (2.4), (2.5), (2.7) and (2.8) turn out respectively to be

(2.16) $$k_{ji}=-l_{jt}f_i{}^t,\quad l_{ji}=k_{jt}f_i{}^t,$$

(2.17) $$k_{jt}p^t=0,\quad l_{jt}p^t=0,\quad k=l=0,$$

(2.18) $$k_{jt}l_i{}^t+k_{it}l_j{}^t=0,$$

(2.19) $$k_{jt}k_i{}^t=l_{jt}l_i{}^t.$$

Thus, (2.15) is reduced to

(2.20) $$A_{ji}=2(l_j{}^tk_{it}+cf_{ji}),$$

where we have put $A_{ji}=\nabla_j n_i-\nabla_i n_j$. By the properties of the almost contact metric structure (f, g, p) induced on M, it follows from (2.17) and (2.20) that

(2.21) $$A_{jr}p^r=0.$$

Applying $f_k{}^i$ to (2.20) and making use of (2.16), we find

(2.22) $$A_{jr}f_i{}^r-2k_{jr}k_i{}^r=2c(g_{ji}-p_jp_i),$$

which implies

(2.23) $$A_{jr}f_i{}^r=A_{ir}f_j{}^r.$$

For the sake of brevity, a tensor $T_{ji}{}^m$ and a function T_m on M for any positive integer m are introduced as follows:

$$T_{ji}{}^m=T_{ji_1}T_{i_2}{}^{i_1}\cdots T_i{}^{i_{m-1}},\quad T_m=\sum_i T_{ii}{}^m.$$

Using this notation, we have from (2.18) and (2.19)

(2.24) $$k_{2m-1}=l_{2m-1}=0,\quad k_{2m}=l_{2m},$$

(2.25) $$k_{ji}{}^m l^{ji}=0,\quad l_{ji}{}^m k^{ji}=0.$$

for any positive integer m.

Since the fact that the distinguished normal C is parallel in the normal bundle is assumed, we see from (2.18) that

$$k_j{}^r\nabla_k l_{ir}+(\nabla_k k_{jr})l_i{}^r+l_j{}^r(\nabla_k k_{ir})+(\nabla_k l_{jr})k_i{}^r=0,$$

which together with (2.11) and (2.12) yield

$$k_j{}^r\nabla_i l_{kr}-k_i{}^r\nabla_j l_{kr}+l_j{}^r\nabla_i k_{kr}-l_i{}^r\nabla_j k_{kr}=0.$$

Therefore, the last two equations give

$$k_j{}^r\nabla_k l_{ir}+l_j{}^r\nabla_k k_{ir}=0,$$

which together with (2.16) imply that

(2.26) $$k_j{}^r \nabla_k k_{ir} - l_j{}^r \nabla_k l_{ir} = 0.$$

3. Parallel normal curvature

Let M be a real $(2n+1)$-dimensional semi-invariant submanifold of codimension 3 in $\overline{M}(c)$. In the sequel we suppose that the normal curvature tensor $R^\perp$ of M in the normal bundle is parallel, namely $\nabla_j R^\perp = 0$. Then we have from the Ricci equation

$$\nabla_i \{\nabla_k n_j - \nabla_j n_k + l_k m_j - l_j m_k\} = 0.$$

Now we suppose that the distinguished normal C is parallel in the normal bundle. Then we have $\nabla_k A_{ji} = 0$ with the aid of the above equation. Thus, differentiating (2.21) covariantly along M and taking account of (2.3), we find $A_{jr} h_{kt} f^{rt} = 0$, which together with (2.23) give

(3.1) $$A_{jr} h_i{}^r = 0$$

because the set (f, g, p) defines an almost contact metric structure.

Hence the equation (2.20) implies

(3.2) $$k_{ir} l_s{}^r h_j{}^s + c f_{ri} h_j{}^r = 0.$$

By the way, since $\nabla_j{}^\perp C = 0$ is assumed, it follows from (2.13) and (2.14) that k_{ji} and l_{ji} are commutative with h_{ji} each other. Thus the last equation yields

(3.3) $$c(h_{jr} f_i{}^r + h_{ir} f_j{}^r) = 0$$

by means of (2.18), which implies

(3.4) $$c(h_{ir} p^r - \alpha p_i) = 0,$$

where we have put $\alpha = h_{rs} p^r p^s$. Differentiating this covariantly and making use of (2.3) we obtain

$$c\{(\nabla_j h_{ir}) p^r - h_{ir} h_{js} f^{rs} - \alpha_j p_i + \alpha h_{jr} f_i{}^r\} = 0,$$

which together (2.10) with $l_j = m_j = 0$ and (3.3) give

(3.5) $$c\{c f_{jk} + h_j{}^r h_{sr} f_k{}^s - \alpha h_{jr} f_k{}^r - \frac{1}{2}(\alpha_k p_j - \alpha_j p_k)\} = 0,$$

where $\alpha_j = \nabla_j \alpha$. By the properties of the almost contact metric structure, it follows that

(3.6) $$c\{c f_{jk} + h_j{}^r h_{sr} f_k{}^s - \alpha h_{jr} f_k{}^r\} = 0,$$

(3.7) $$c(\alpha_j - B p_j) = 0$$

for some function B on M. Hence (3.1) and (3.6) give rise to $c^2 A_{jr} f_i{}^r = 0$. Consequently we have $c^2 A_{ji} = 0$ because the set (f, g, p)

defines an almost contact metric structure. Thus, relationship (2.22) gives

(3.8) $$c^2\{k_{ji}^2+c(g_{ji}-p_jp_i)\}=0$$

and hence $c^2(2nc+k_2)=0$. Therefore we have following fact:

PROPOSITION 3.1. *Let M be a semi-invariant submanifold of codimension 3 in a complex space form $\overline{M}(c)$, $(c\geqq 0)$. If the distinguished normal C and the normal curvature of M are parallel in the normal bundle, then the ambient space is Euclidean.*

Now we suppose that c does not vanish. Then (3.6) and (3.7) reduce respectively to

(3.9) $$h_{ji}^2=\alpha h_{ji}+c(g_{ji}-p_jp_i),$$

$\alpha_j=Bp_j$ because of (3.7). Differentiation (3.7) covariantly yields $\nabla_k\alpha_j=(\nabla_kB)p_j-Bh_{kr}f_j{}^r$, which implies $(\nabla_kB)p_j-(\nabla_jB)p_k+2Bh_{jr}f_k{}^r=0$, where we have used (2.3) and (3.3). ∇_kB is proportional to p_k, it follows that $Bh_{jr}f_k{}^r=0$, which means $B(h_{ji}-\alpha p_jp_i)=0$. From this fact and (3.9) we can see that α is constant on M. Therefore (3.9) means that $h_j{}^i$ has constant eigenvalues. By differentiating (3.9) and taking account of (2.3), we get

$$(\nabla_kh_{jr})h_i{}^r+h_j{}^r(\nabla_kh_{ir})-\alpha\nabla_kh_{ji}=c\{(h_{kr}f_j{}^r)p_i+(h_{kr}f_i{}^r)p_j\}.$$

If we make use of (2.3), (2.10) with $l_j=m_j=0$, (3.3) and (3.4), then the covariant derivative of the second fundamental form h_{ji} is given by (for detail, see [4], [5])

(3.10) $$\nabla_kh_{ji}=c(f_{ik}p_j+f_{jk}p_i).$$

On the other hand, we have from the Gauss equation (2.9) that the Ricci tensor of the semi-invariant submanifold is obtained:

$$R_{ji}=c\{(2n+3)g_{ji}-3p_jp_i\}-2k_{ji}^2+hh_{ji}-h_{ji}^2,$$

where $h=\Sigma_ih_{ii}$ because of (3.8) and (3.9), which reduces to

(3.11) $$R_{ji}=2c\{(n+2)g_{ji}-2p_jp_i\}+(h-\alpha)h_{ji}.$$

Differentiating (3.11) covariantly and using (2.3) and (3.10), we find

$$\nabla_kR_{ji}=4c\{(h_{kr}f_j{}^r)p_i+(h_{kr}f_i{}^r)p_j\}+c(h-\alpha)(f_{ik}p_j+f_{jk}p_i)$$

because of the facts that α is constant and eigenvalues of $h_j{}^i$ are constant.

If we suppose that M has harmonic curvature, that is, $\nabla_kR_{ji}-\nabla_jR_{ki}$

$=0$, then we see from the last equation that

$$4c\{2(h_{kr}f_j{}^r)p_i+(h_{kr}f_i{}^r)p_j-(h_{jr}f_i{}^r)p_k\}$$
$$+c(h-\alpha)\{f_{ik}p_j-f_{ij}p_k+2f_{jk}p_i\}=0.$$

Transforming this by p^j and taking account of (3.4), we get $c\{4h_{kr}f_i{}^r+(h-\alpha)f_{ik}\}=0$ and hence $h-\alpha=0$ because of $c\neq 0$. Thus, it follows that $h_{ji}=\alpha p_j p_i$. But this is impossible because of (3.9). Therefore, we have

PROPOSITION 3.2. *Let M be a semi-invariant submanifold of codimension 3 in a complex space form $\overline{M}(c)$. Suppose that the distinguished normal and the normal curvature are parallel in the normal bundle. If M has harmonic curvature, then the ambient space is Euclidean.*

Now we put

(3.12) $$D_k k_{ji}=\nabla_k k_{ji}-n_k l_{ji},\ D_k l_{ji}=\nabla_k l_{ji}+n_k k_{ji},$$

then using (2.11) and (2.12) we can easily see that $D_k k_{ji}$ and $D_k l_{ji}$ are symmetric for all indices provided that the distinguished normal is parallel in the normal bundle.

4. Semi-invariant submanifolds in a complex Euclidean space

In this section we consider a semi-invariant submanifold M of codimension 3 in a Euclidean $2(n+2)$-space such that the distinguished normal C and the normal curvature of M are parallel in the normal bundle. Then (3.2) becomes to $k_{ir}l_s{}^r h_j{}^s=0$ and hence

(4.1) $$k_{jr}h_i{}^r=0,\ l_{jr}h_i{}^r=0$$

because of (2.18) and (2.19). Furthermore (2.20) turns out to be $2k_{ji}{}^2=A_{jr}f_i{}^r$. Since the fact that the normal curvature of M is parallel in the normal bundle is assumed, it follows, in a direct consequence of (3.1), that

(4.2) $$(\nabla_k k_{jr})k_i{}^r+(\nabla_k k_{ir})k_j{}^r=0.$$

If we take the skew-symmetric part of this with respect to the indices k and j and make use of (2.11) and (3.12), then we obtain $(n_k l_{jr}-n_j l_{kr})k_i{}^r+(D_i k_{kr}+n_k l_{ir})k_j{}^r-(D_i k_{jr}+n_j l_{ir})k_k{}^r=0$ and consequently $(D_i k_{kr})k_j{}^r-(D_i k_{jr})k_k{}^r=0$ with the aid of (2.18). Combining this with (4.2), we have

$$k_{jr}D_ik_k{}^r=0, \tag{4.3}$$

where we have used (2.18) and (3.12). In the same way we have

$$l_{jr}D_il_k{}^r=0. \tag{4.4}$$

Using the last two equations and making use of (2.25), it is easily seen that k_{2m} and l_{2m} are constant for any positive integer m. If we take account of (2.20) with $c=0$, then the equation (4.3) leads to

$$k_j{}^r\nabla_ik_{kr}=\frac{1}{2}n_iA_{kj}.$$

Differentiating this covariantly, we find

$$(\nabla^lk_j{}^r)(\nabla_ik_{kr})+k_j{}^r\nabla^l\nabla_ik_{kr}=\frac{1}{2}(\nabla_ln_i)A_{kj}$$

since the normal curvature of M is parallel. If we take the skew-symmetric part of this with respect to indices l and i and make use of the Ricci identity, then we obtain

$$(\nabla_lk_j{}^r)(\nabla_ik_{kr})-(\nabla_ik_j{}^r)(\nabla_lk_{kr}) \tag{4.5}$$
$$=R_{lik}{}^rk_{jr}{}^2+R_{lirs}k_k{}^sk_j{}^r+\frac{1}{2}A_{li}A_{kj}.$$

Thus the symmetric part of this with respect to the indices j and k gives

$$R_{lik}{}^rk_{jr}{}^2+R_{lij}{}^rk_{kr}{}^2=0,$$

which together with the Gauss equation of M in a Euclidean space yield

$$k_{lj}{}^3k_{ik}-k_{lk}k_{ij}{}^3+l_{lj}{}^3l_{ik}-l_{lk}l_{ij}{}^3$$
$$+k_{lk}{}^3k_{ij}-k_{lj}k_{ik}{}^3+l_{lk}{}^3l_{ij}-l_{lj}l_{ik}{}^3=0,$$

where we have used (2.18), (2.19) and (4.1).

Transforming the last expression by $k_i{}^k$ and using (2.24) and (2.25), then one can get

$$k_2k_{jl}{}^3-k_4k_{jl}=0. \tag{4.6}$$

In the same way we see from (4.4) that

$$k_2l_{jl}{}^3-k_4l_{jl}=0. \tag{4.7}$$

If we suppose that the constant k_2 does not zero on M, then (4.6) and (4.7) reduces respectively to

$$k_{ji}{}^3=Ak_{ji},\quad l_{ji}{}^3=Al_{ji}. \tag{4.8}$$

for a constant A given by $A=k_4/k_2$. Differentiating the first equation of (4.8) covariantly and making use of (4.2), we find $A\nabla_kk_{ji}=(\nabla_kk_j{}^r)k_{ir}{}^2$, or, using (4.3) and (4.8) it follows that $AD_kk_{ji}=0$ and hence D_kk_{ji}

$=0$. Thus the equation (4.5) turns out to be

$$R_{lik}{}^{r}k_{jr}{}^{2}+R_{lirs}k_{k}{}^{s}k_{j}{}^{r}+\frac{1}{2}A_{li}A_{kj}=0.$$

If we substitute (2.9) and (2.20) with $c=0$ into the last equation and take account of (4.1), then we get

$$k_{l}{}^{r}k_{ik}\ -k_{lk}k_{i}{}^{r}+l_{l}{}^{r}l_{ik}-l_{lk}l_{i}{}^{r})k_{jr}{}^{2}$$
$$+(k_{l}{}^{s}k_{i}{}^{r}-k_{l}{}^{r}k_{i}{}^{s}+l_{l}{}^{s}l_{i}{}^{r}-l_{l}{}^{r}l_{i}{}^{s})k_{jr}k_{ks}-2k_{lr}l_{i}{}^{r}k_{js}l_{k}{}^{s}=0.$$

Thus, by contracting j and l and using (2.18), (2.19) and (4.8) we can verify that

$$2k_{ki}{}^{4}+2Ak_{ki}{}^{2}+k_{2}k_{ik}{}^{2}=0,$$

which implies $k_{2}=0$. Thus we have $k_{ji}=0$ on M and hence $l_{ji}=0$ because of (2.19). Therefore we have

LEMMA 4.1. *Let M be a semi-invariant submanifold of codimension 3 in a Euclidean $2(n+2)$-space. If the distinguished normal C and the normal curvature of M are parallel in the normal bundle, then $k_{ji}=l_{ji}=0$.*

According to Proposition 3.2 and Lemma 4.1 we prove the following fact:

THEOREM 4.2. *Let M be a simply connected complete semi-invariant submanifold of codimension 3 in a real $2(n+2)$-dimensional complex space form $\overline{M}(c)$. Suppose that the distinguished normal C and the normal curvature of M are parallel in the normal bundle. If M has harmonic curvature, then $c=0$. In particular, if $c=0$ and the shape operator in the direction of the distinguished normal has no simple roots, then M is isometric one of the following spaces:*

$$E^{2n+1},\quad S^{2n+1}\ \text{or}\ S^{r}\times E^{2n-r+1}.$$

Proof. According to Proposition 3.2, the ambient space is Euclidean. Thus, the Ricci tensor of M is given by $R_{ji}=hh_{ji}-h_{ji}{}^{2}$ because of Lemma 4.1. Since M has harmonic curvature, the above equation implies

$$h_{k}h_{ji}-h_{j}h_{ki}=\nabla_{k}h_{ji}{}^{2}-\nabla_{j}h_{ki}{}^{2}, \tag{4.9}$$

where $h_{k}=\nabla_{k}h$. Since the fact that the shape operator in the direction of C has no simple roots is assumed, it is well known from (4.9) that the mean curvature of M is constant, namely $h_{k}=0$ (for detail see [6], [7]). Thus (4.9) means that $h_{ij}{}^{2}$ is of Codazzi type and $\nabla_{k}h_{ji}=0$ (cf.

[6], [7], [9]). Thus the first normal space $N_1(p)$ defined to be orthogonal complement of $\{C_p \in N_p(M) : H_{C_p}=0\}$ in $N_p(M)$ is invariant under the parallel translation and of constant dimension 1, where H_{C_p} are the second fundamental forms associated C_p and $N_p(M)$ is the normal space at p in M. Thus, by the reduction theorem [3], we conclude that there exists a $2(n+1)$-dimensional totally geodesic submanifold E^{2n+2} in E^{2n+4} in which M is hypersurface with parallel second fundamental form. Since M is complete and simply connected, due to [8], we see that M is isometric with a plane E^{2n+1}, a sphere S^{2n+1} or $S^r \times E^{2n-r+1}$. This completes the proof.

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