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SURJECTIVITY OF GENERALIZED LOCALLY EXPANSIVE MAPS

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1. Introduction

Browder [1] established some fundamental surjectivity theorems on a map T of a Banach space E into a Banach space F in which the hypotheses on T are completely local in character. He proved that if T is a locally expansive, continuous open map of E into F, then T is a homeomorphism onto F [1].

In 1979, Kirk and Schönberg [3] proved the following generalized version of Browder's result :

THEOREM [3]. *Let X and Y be complete metric spaces with Y metrically convex, and $T: X \longrightarrow Y$ an open map having closed graph. Suppose also that T is locally expansive on X. Then $T(X)=Y$.*

Here, a map T of a metric space X into a metric space Y is said to have closed graph if $x_n \longrightarrow x$ in X and $Tx_n \longrightarrow y$ in Y implies $Tx=y$. And following Menger [4], a metric space Y is said to be metrically convex if for all u, v in Y with $u \neq v$ there exists $w \in Y$ distinct from u and v, such that

$$d(u, v)=d(u, w)+d(w, v).$$

In section 2, we apply the Maximal Ordering Principle of Turinici to more generalized version of the lemma of Kirk and Schönberg which was proved by using a continuation method.

In section 3, we prove a surjectivity theorem for locally c-expansive maps, and obtain some related results.

2. Maximal Element Techniques

In [6], Turinici introduced the Maximal Ordering Principle. In order

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to state the principle we need the following terms :

Let X be a metric space and $\leq$ an order on X. A subset $D\subset X$ is said to be order-closed if for every monotone increasing sequence $\{x_n| n\in N\}$ in D and every $x\in X$ with $x_n\rightarrow x$ as $n\rightarrow\infty$ we have $x\in D$, and the ambient order $\leq$ on X is said to be self-closed if $S(x)=\{u\in X| u\geq x\}$ is order-closed for all $x\in X$. Finally, the ambient metric space (X, d) is said to be order-compact if every monotone increasing sequence in X has a convergent (monotone) subsequence.

Now we state the Maximal Ordering Principle of Turinici :

THEOREM [6]. *Suppose that the metric space (X, d) and the order $\leq$ on X are such that*

(1) *$\leq$ is self-closed on X,*

(2) *(X, d) is order-compact.*

Then, for every $x\in X$, there exists a maximal element $z\in X$ such that $x\leq z$.

We apply the Maximal Ordering Principle of Turinici to more generalized version of the lemma of Kirk and Schönberg [3].

LEMMA. *Let X be a metric space, Y a complete metric space with metric convexity and $T:\bar{B}\longrightarrow Y$ an open map having closed graph where B is open in X. Suppose that*

(*a*) *T is injective on $\bar{B}$, and*

(*b*) *for any Cauchy sequence $\{Tv_n\}$ in Y, $\{v_n\}$ is relatively compact. Suppose also that for fixed $y\in Y$, there exists u_0 in $\bar{B}$ such that*

$$d(Tu_0, y)\geq d(Tu, y)+d(Tu_0, Tu)$$

for all $u\in\partial B$.

Then there exists $u\in B$ such that $Tu=y$.

Proof. We consider a relation $\leq$ on $\bar{B}$ defined by for any $u, v\in\bar{B}$, $u\leq v$ iff

$$d(Tu, y)\geq d(Tv, y)+d(Tu, Tv) \text{ for a fixed } y\in Y.$$

Then it is easy to see that the relation $\leq$ is reflexive, antisymmetric and transitive, i. e., $\leq$ is actually an order on $\bar{B}$.

So, in order to apply the principle of Turinici, we claim that :

(1) $S(u)$ is order-closed for any $u\in\bar{B}$, and

(2) $(\bar{B}, d)$ is order-compact.

To prove (1), we choose any monotone increasing sequence $\{v_n\}$ in

$S(u)$. Since $v_n \leq v_{n+1}$, for a fixed $y \in Y$, we have

$$d(Tv_n, y) \geq d(Tv_{n+1}, y) + d(Tv_n, Tv_{n+1}).$$

Hence $\{Tv_n\}$ is Cauchy. Since T satisfies (b), a subsequence $\{u_{n_i}\}$ is convergent in $\bar{B}$. Furthermore, for all n,

$$d(Tu, y) \geq d(Tv_n, y) + d(Tu, Tv_n).$$

Since T has closed graph, $Tv_{n_i} \longrightarrow Tv$ in Y for some v in $\bar{B}$, and

$$d(Tu, y) \geq d(Tv, y) + d(Tu, Tv).$$

Hence $v \in S(u)$. This completes the proof of (1).

And any monotone increasing sequence $\{v_n\}$ in $\bar{B}$ has a convergent subsequence. Indeed, since $\{Tv_n\}$ is a Cauchy sequence, by (a), $\{v_n\}$ has a convergent subsequence. This proves (2).

Thus by the principle, we have a maximal element v_0 in $\bar{B}$ such that $u_0 \leq v_0$ and

$$d(Tu_0, y) \geq d(Tv_0, y) + d(Tv_0, Tu_0).$$

From the boundary condition, we have $v_0 \in B$. Since $T(B)$ is open and $T(v_0) \in T(B)$, if $y \notin T(B)$, then by Menger's theorem [4], there exists $T(u) \neq T(v_0)$ for some $u \in B$ such that

$$d(Tv_0, y) \geq d(Tu, y) + d(Tv_0, Tu).$$

Hence there exists $u \in B$ such that $v_0 < u$. This is a contradiction to the maximality of v_0. So $y \in T(B)$.

3. Surjectivity of Generalized Locally Expansive Maps

For a Banach space, we may consider more generalized classes of maps than those of locally expansive ones.

Let $c : [0, \infty) \longrightarrow (0, \infty)$ be a continuous nonincreasing function such that $\int_0^\infty c(s)\,ds = \infty$.

For convenience, a nonlinear map T from a subset B of a Banach space X into a metric space Y is said to be locally c-expansive if each $x \in B$ has a neighborhood N of x in B such that

$$c(\|x\|)\,\|u-v\| \leq d(Tu, Tv) \text{ for all } u, v \in N.$$

From the lemma in section 2, we obtain the following surjectivity theorem of locally c-expansive maps :

THEOREM. *Let X be a Banach space, Y a complete metric space with* *metric convexity, and* B *open in* X. *Let* $T:\bar{B}\longrightarrow Y$ *have closed graph. If* T *is locally* c*-expansive and open on* B, *then for* $y\in Y$ *the following are equivalent:*

(*a*) $y\in T(B)$.

(*b*) *there exists* $x_0\in B$ *such that* $d(Tx_0, y)\leq d(Tx, y)$ *for all* $x\in\partial B$.

Proof. (a)$\Longrightarrow$(b) is trivial.

(b)$\Longrightarrow$(a). For given $u\in B$, we let $r(u)$ denote the supremum of all $r\in[0,1]$ such that $B(u,r)\subset B$ and $c(\|u\|)\|u_1-u_2\|\leq d(Tu_1, Tu_2)$ for all $u_1, u_2\in B(u,r)$ where $B(u,r)$ denotes the open ball of radius r around u. Since B is open and T is locally c-expansive on B, $r(u)>0$ for all $u\in B$. Furthermore, by definition, $\bar{B}(u, r(u)/2)\subset B$ and $c(\|u\|)\|u_1-u_2\|\leq d(Tu_1, Tu_2)$ for all $u_1, u_2\in\bar{B}(u, r(u)/2)$ where $\bar{B}(u, r(u)/2)$ denotes the closed ball of radius $r(u)/2$ around u. Assume, on the contrary, that $y\notin T(B)$. Then the negation of the above lemma implies the existence of a sequence $\{u_n\}$ in B such that the following four conditions hold :

(1) $u_1=x_0$;

(2) $c(\|u_n\|)\|u_{n+1}-u_n\|\leq d(Tu_{n+1}, Tu_n)$ for all n;

(3) $\|u_{n+1}-u_n\|=r(u_n)/2$ for all n ;

(4) $d(Tu_{n+1}, y)+d(Tu_{n+1}, Tu_n)\leq d(Tu_n, y)$ for all n.

Then (1) and (4) imply by induction

$$(5)\quad d(Tu_{n+1}, y)+\sum_{j=1}^{n} d(Tu_{j+1}, Tu_j)\leq d(Tx_0, y)$$

for all n.

In particular, $\sum_{j=1}^{\infty} d(Tu_{j+1}, Tu_j)<\infty$ and by (2) $\sum_{j=1}^{\infty} c(\|u_j\|)\|u_{j+1}-u_j\|<\infty$. We claim that $\{\|u_n\|\}$ is bounded. On the contrary, we assume that $\{\|u_n\|\}$ is unbounded. Then we may choose a subsequence $\{u_{j_k}\}$ such that

(6) $\|u_{j_1}\|<\|u_{j_2}\|<\cdots$, $\lim_k\|u_{j_k}\|=\infty$, and $1=j_1<j_2<\cdots$;

(7) if $j_k<l<j_{k+1}$, then $\|u_l\|\leq\|u_{j_k}\|$.

Then for $k=1,2,3,\cdots$,

$$\begin{aligned} c(\|u_{j_k}\|)(\|u_{j_{k+1}}\|-\|u_{j_k}\|) &\leq c(\|u_{j_k}\|)\|u_{j_{k+1}}-u_{j_k}\| \\ &\leq c(\|u_{j_k}\|)(\|u_{j_k}-u_{j_k+1}\|+\|u_{j_k+1}-u_{j_k+2}\|+\cdots+\|u_{j_{k+1}-1}-u_{j_{k+1}}\|) \\ &\leq c(\|u_{j_k}\|)\|u_{j_k}-u_{j_k+1}\|+c(\|u_{j_k+1}\|)\|u_{j_k+1}-u_{j_k+2}\| \end{aligned}$$

$$+\cdots+c(\|u_{j_{k+1}-1}\|)\|u_{j_{k+1}-1}-u_{j_{k+1}}\|.$$

Since $\int^{\infty}c(s)ds=\infty$,

$$\infty=\sum_{k=1}^{\infty}c(\|u_{j_k}\|)\,\|u_{j_{k+1}}\|-\|u_{j_k}\|)$$
$$\leq\sum_{k=1}^{\infty}c(\|u_j\|)\,\|u_{j+1}-u_j\|<\infty.$$

This is a contradiction. Hence $\{\|u_n\|\}$ is bounded and $\{u_n\}$ is Cauchy. Since X and Y are complete, there exists $x\in\bar{B}$ such that $u_n\longrightarrow x$ and $Tu_n\longrightarrow y'$ as $n\rightarrow\infty$. By the assumption that T has closed graph, we have $y'=Tx$. Since $r(u_n)\longrightarrow 0$, $x\notin B$, i.e., $x\in\partial B$. Since $\{\|u_n\|\}$ is bounded, $\alpha=\inf c(\|u_n\|)$ is a positive number. Hence (2) and (5) yield for all n,

$$\begin{aligned}&d(Tu_{n+1},y)+\alpha\|u_{n+1}-x_0\|\\&\leq d(Tu_{n+1},y)+\alpha\sum_{j=1}^{n}\|u_{j+1}-u_j\|\\&\leq d(Tu_{n+1},y)+\sum_{j=1}^{n}c(\|u_j\|)\,\|u_{j+1}-u_j\|\\&\leq d(Tu_{n+1},y)+\sum_{j=1}^{n}d(Tu_{j+1},Tu_j)\\&\leq d(Tx_0,y),\end{aligned}$$

so that letting $n\rightarrow\infty$ we get

$$d(Tx,y)+\alpha\|x-x_0\|\leq d(Tx_0,y).$$

Since $x\in\partial B$, this contradicts (b).

Taking $B=X$ in the above theorem, we obtain the following :

COROLLARY 1. *Let X be a Banach space and Y a complete metric space with metric convexity. Let T: $X\rightarrow Y$ have closed graph. If T is open and locally c-expansive, then $T(X)=Y$.*

The above corollary can be obtained from the surjectivity theorem of Ray and Walker [5] if Y is a Banach space. Also Theorem 3.4 in [5] can be obtained from the above corollary.

Here we need the following terms : Let X^* denote the dual of a real Banach space X. The duality map J from X into 2^{X^*} is defined by

$$J(x)=\{j\in X^*\mid(x,j)=\|x\|^2\text{ and }\|j\|=\|x\|\}.$$

It is well-known that, by the Hahn-Banach theorem, $J(x)$ is not empty for each $x \in X$, J is single-valued when X^* is strictly convex and J is uniformly continuous on bounded subsets of X whenever X^* is uniformly convex [2].

COROLLARY 2 [5, Theorem 3. 4]. *Let X be a Banach space, P a continuous selfmap of X, and $c : [0, \infty) \longrightarrow [0, \infty)$ a continuous nondecreasing function for which $\int^{\infty} c(s)ds = \infty$. Suppose also that for each $x, y \in X$ there exists a $j \in J(x-y)$ for which*

$$(Px-Py, y) \geq c(\max \{\|x\|, \|y\|\})\ \|x-y\|^2.$$

Then P is a homeomorphism on X.

Proof. In [5] Ray and Walker obtained 1 domain invariance result on P, i. e., P is an open map. By Corollary 1 we can show that P is surjective. Fix $\varepsilon > 0$. Let $\bar{c}(r) = c(r+\varepsilon)$. Then for any $x \in X$, x_1, $x_2 \in B(x, \varepsilon)$, we have

$$\begin{aligned} \|Px_1 - Px_2\| &\geq c(\max(\|x_1\|, \|x_2\|)\|x_1 - x_2\| \\ &\geq c(\|x\| + \varepsilon)\|x_1 - x_2\| \\ &\geq \bar{c}(\|x\|)\|x_1 - x_2\|. \end{aligned}$$

Hence P satisfies the hypotheses of Corollary 1. Therefore P is surjective and, hence, a homeomorphism from X onto X.

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