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ASYMPTOTIC BEHAVIOR OF SEMIGROUPS OF ASYMPTOTICALLY NONEXPANSIVE TYPE ON BANACH SPACES

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1. Introduction

Let G be a semitopological semigroup. G is called right reversible if any two closed left ideals of G has non-void intersection. In this case, $(G, \geqq)$ is a directed system when the binary relation "$\geqq$" on G is defined by $t \geqq s$ if and only if $\{s\} \cup \overline{Gs} \supseteq \{t\} \cup \overline{Gt}$, $s, t \in G$. Right reversible semitopological semigroups include all commutative semigroups and all semitopological semigroups which are right amenable as discrete semigroups (see [9]). Left reversibility of G is defined similarly. G is called reversible if it is both left and right reversible.

In 1976, Kirk [12] introduced any non-Lipschitzian self-mapping which extends, in a sense, an asymptotically nonexpansive mapping inherited by Goebel and Kirk [4] ; a continuous mapping $T : K \to K$, K a nonempty closed subset of a real Banach space X, is said to be of asymptotically nonexpansive type if for each $x \in K$,

$$\limsup_{n\to\infty} \{\sup[\|T^n x - T^n y\| - \|x-y\|] : y \in K\} \leqq 0.$$

Now, we introduce a semigroup of non-Lipschitzian self-mappings ; let C be a nonempty closed convex subset of a real Banach space X with norm $\|\cdot\|$. A family $\mathscr{T} = \{T_s : s \in G\}$ of continuous mappings of C into C is said to be a right reversible semigroup of asymptotically nenexpansive type on C if the following conditions are satisfied :

(a) the index set G is a right reversible semitopological semigroup with the above order $\geqq$;

(b) $T_{st}x = T_s T_t x$ for all $s, t \in G$ and $x \in C$;

(c) for each $x \in C$,

$$\limsup_{s\in G} \{\sup[\|T_s x - T_s y\| - \|x-y\|] : y \in C\} \leqq 0 ;$$

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(d) T is continuous with respect to the strong operator topology : $T_s x \to T_t x$ for each $x \in C$ as $s \to t$ in G.

Left reversible semigroup of asymptotically nonexpansive type is defined similarly. For semigroups of another non-Lipschitzian self-mappings, see [3], [10], [11] etc.

For each $x \in C$, $\mathscr{T}(x) = \{T_s x : s \in G\}$ is called the orbit of x under $\mathscr{T}$ and a point $z \in C$ such that $\mathscr{T}(z) = \{z\}$ is called a common fixed point of $\mathscr{T}$. We denote by $F(\mathscr{T})$ the set of common fixed points of $\mathscr{T}$ and by $\omega_w(x)$ the set of weak subnet limits of the net $\{T_s x : s \in G\}$ and set $E(x) = \{y \in C : \lim_{s \in G} \|T_s x - y\| \text{ exists}\}$.

It is the purpose of this paper that some of the weak convergence and fixed point theory of semigroups of nonexpansive mappings ([8], [13], [14]) carries over to the larger class of mappings definded above.

2. Weak convergence

Unless other specified, let $G, X, C, \mathscr{T} = \{T_s : s \in G\}$ be as before. We begin with the following

LEMMA 2.1. *For each $x \in C$, $F(\mathscr{T}) \subseteq E(x)$.*

Proof. Let $y \in F(\mathscr{T})$ and $r = \inf_{s \in G} \|T_s x - y\|$. Given $\varepsilon > 0$, there is $s_0 \in G$ such that $\|T_{s_0} x - y\| < r + \frac{\varepsilon}{2}$. Since $\mathscr{T}$ is of asymptotically nonexpansive type, there also exists $t_0 \in G$ such that

$$\|T_t T_{s_0} x - y\| \leqq \|T_{s_0} x - y\| + \frac{\varepsilon}{2}$$

for all $t \geqq t_0$. Let $b \geqq a_0 = t_0 s_0$. Since G is right reversible, we may assume $b \in \overline{Ga_0}$. Let $\{s_\alpha\}$ be a net in G such that $s_\alpha a_0 \to b$. Then, for each α,

$$\|T_{s_\alpha t_0} T_{s_0} x - y\| \leqq \|T_{s_0} x - y\| + \frac{\varepsilon}{2}$$

Hence $\|T_b x - y\| \leqq \|T_{s_0} x - y\| + \frac{\varepsilon}{2}$. So, we have

$$\inf_s \sup_{t \geqq s} \|T_t x - y\| \sup_{b \geqq a_0} \|T_b x - y\| \leqq \|T_{s_0} x - y\| + \frac{\varepsilon}{2} < r + \varepsilon.$$

Since ε is arbitrary, we have $\inf_s \sup_{t \geqq s} \|T_t x - y\| \leqq r = \inf_{s \in G} \|T_s x - y\|$.

Therefore, $\lim_s \|T_s x - y\|$ exists and so $y \in E(x)$.

LEMMA 2.2. *Let X be uniformly convex and suppose that $F(\mathfrak{T})\neq\emptyset$. Let $x\in C$, $f\in F(\mathfrak{T})$ and $0<\alpha\leqq\beta<1$. Then, for each $\varepsilon>0$, there is $a_0\in G$ such that*

$$\|T_s(\lambda T_t x+(1-\lambda)f)-(\lambda T_s T_t x+(1-\lambda)f)\|<\varepsilon$$

for all s, $t\in G$ with $s,t\geqq a_0$ and $\lambda:\alpha\leqq\lambda\leqq\beta$.

Proof. Let $\varepsilon>0$, $c=\min\{2\lambda(1-\lambda):\alpha\leqq\lambda\leqq\beta\}$, $c'=\max\{2\lambda(1-\lambda):\alpha\leqq\lambda\leqq\beta\}$ and $r=\lim_s\|T_s x-f\|$. For $r=0$, it is easy. Let $r>0$. Then we can choose $d>0$ so small that

$$(r+d)\left[1-c\delta\left(\frac{\varepsilon}{r+d}\right)\right]<r,$$

where δ is the modulus of convexity of the norm. Since $r=\lim_s\|T_s x-f\|$ and $\mathfrak{T}$ is of asymptotically nonexpansive type, there exists $a_0\in G$ such that

$$\|T_s x-f\|<r+\frac{d}{2},\quad \|f-T_s z\|<\frac{c}{4}d+\|f-z\|$$

and

$$\|T_s T_t x-T_s z\|<\frac{c}{4}d+\|T_t x-z\|$$

for all $s\geqq a_0$, $z\in C$ and each $t\in G$. Suppose that

$$\|T_s(\lambda T_t x+(1-\lambda)f)-(\lambda T_s T_t x+(1-\lambda)f)\|\geqq\varepsilon$$

for some $s,t\geqq a_0$ and $\lambda:\alpha\leqq\lambda\leqq\beta$. Put $u=(1-\lambda)(T_s z-f)$ and $v=\lambda(T_s T_t x-T_s z)$, where $z=\lambda T_t x+(1-\lambda)f$. Then we have that $\|u\|$, $\|v\|<\lambda(1-\lambda)(r+d)$, $\|u-v\|=\|T_s z-(\lambda T_s T_t x+(1-\lambda)f)\|\geqq\varepsilon$ and $\lambda u+(1-\lambda)v=\lambda(1-\lambda)(T_s T_t x-f)$. So, by using the Lemma in [7] we have

$$\|T_{st}x-f\|\leqq(r+d)\left[1-c\delta\left(\frac{\varepsilon}{r+d}\right)\right]<r.$$

This contradicts $r=\inf_s\|T_s x-f\|$ by Lemma 2.1.

Let x and y be elements of a Banach space X. Then we denote by $[x,y]$ the set $\{\lambda x+(1-\lambda)y:0\leqq\lambda\leqq1\}$ and $\overline{\mathrm{co}}(A)$ denotes the closure of the convex hull of A.

LEMMA 2.3. *Let X have a Fréchet differentiable norm and $\{x_\alpha\}$ a bounded net in C. Let $z\in\bigcap_\beta\overline{co}\{x_\alpha:\alpha\geqq\beta\}$, $y\in C$ and $\{y_\alpha\}$ a net of elements in C with $y_\alpha\in[y,x_\alpha]$ and $\|y_\alpha-z\|=\min\{\|u-z\|:u\in[y,x_\alpha]\}$.*

If $y_\alpha \to y$, *then* $y=z$.

For the proof of Lemma 2.3, see Lemma 3 in [14]. Now we can prove the following :

Proposition 2.4. *Let* C *be a closed convex subset of a uniformly convex Banach space* X *with a Fréchet differentiable norm. Suppose that* $F(\mathscr{T})$ *is nonempty. Then, for each* $x \in C$, *the set* $\bigcap_{s \in G} \overline{co}\{T_t x : t \geqq s\} \cap F(\mathscr{T})$ *consists of at most one point*

Proof. For each $x \in C$, let $W(x) = \bigcap_t \overline{\text{co}}\{T_t x : t \geqq s\}$. Suppose that $f, g \in W(x) \cap F(\mathscr{T})$ and $f \neq g$. Put $h = \frac{1}{2}(f+g)$ and $r = \lim_s \| T_s x - g\|$. Since $h \in W(x)$, $\|h-g\| \leqq r$. For each $s \in G$, choose $p_s \in [T_s x, h]$ such that $\|p_s - g\| = \min\{\|y-g\| : y \in [T_s x, h]\}$. If $\liminf_s \| p_s - g\| = \|h-g\|$, then obviously $p_s \to h$. Hence, by Lemma 2.3, $h=g$. This contradicts $f \neq g$. Now we suppose that $\liminf_s \|p_s - g\| < \|h-g\|$. Then there exists $c > 0$ and $s_\alpha \in G$ such that $s_\alpha \geqq \alpha$ and

$$\|p_{s_\alpha} - g\| + c < \|h-g\|$$

for every $\alpha \in G$. Put $p_{s_\alpha} = a_\alpha T_{s_\alpha} x + (1-a_\alpha) h$ for every α. Then there is $\beta > 0$ and $\gamma < 1$ such that $\beta \leqq a_\alpha \leqq \gamma$ for all α. By Lemma 2.2, and since $\mathscr{T}$ is of asymptotically nonexpansive type, there exists $\alpha_0 \in G$ such that

$$\| T_s(\lambda T_t x + (1-\lambda) h) - (\lambda T_s T_t x + (1-\lambda) h) \| < \frac{c}{2}$$

and

$$\|g - T_s z\| < \frac{c}{2} + \|g - z\|$$

for all $s, t \geqq \alpha_0$, $z \in C$ and $\lambda : \beta \leqq \lambda \leqq \gamma$.

For $s_{\alpha_0} \geqq \alpha_0$, let $s \geqq \beta_0 = \alpha_0 s_{\alpha_0}$. Then, since G is right reversible, $s \in \{\beta_0\} \cup \overline{G}_{\beta_0}$, we may assume $s \in \overline{G}_{\beta_0}$. Let $\{t_\beta\}$ be a net in G such that $t_\beta \beta_0 \to s$. Then, for each β,

$$\begin{aligned} \|p_{t_\beta \beta_0} - g\| &\leqq \|a_{\alpha_0} T_{t_\beta \beta_0} x + (1-a_{\alpha_0}) h - g\| \\ &\leqq \| T_{t_\beta \alpha_0} p_{s_{\alpha_0}} - (a_{\alpha_0} T_{t_\beta \alpha_0} T_{s_{\alpha_0}} x + (1-a_{\alpha_0}) h\| \\ &\quad + \|g - T_{t_\beta \alpha_0} p_{s_{\alpha_0}}\| \leqq c + \|g - p_{s_{\alpha_0}}\| < \|h-g\|. \end{aligned}$$

Hence, $\|p_s - g\| < \|h-g\|$ for all $s \geqq \beta_0$. Thererfore we have $p_s \neq h$ for all $s \geqq \beta_0$. Let $s \geqq \beta_0$ and $u_k = k(h - T_s x) + T_s x$ for all $k \geqq 1$. Then

$\|u_k-g\| \geqq \|h-g\|$ for all $k \geqq 1$ and hence, by Theorem 2. 5 of [2], we have

$$\langle h-u_k, \ J(g-h)\rangle = \langle (1-k)(h-T_s x), J(g-h)\rangle \geqq 0$$

for all $k \geqq 1$, where J is the duality mapping of X. Therefore, it follows that $\langle h-T_s x, \ J(g-h)\rangle \leqq 0$ for all $s \geqq \beta_0$. Then we have $\langle h-y, \ J(g-h)\rangle \leqq 0$ for all $y \in \overline{\text{co}}\{T_t x : t \geqq \beta_0\}$ Put $y=f=h+(h-g)$, then $h=g$, This contradicts $f \neq g$. The proof is completed.

THEOREM 2. 5. *Let X, C, and $F(\mathcal{T})$ as in Proposition 2. 4. Let $x \in C$. If $\omega_w(x) \subseteq F(\mathcal{T})$, then the net $\{T_s x : s \in G\}$ converges weakly to some $y \in F(\mathcal{T})$.*

Proof. Since $F(\mathcal{T}) \neq \phi$, by Lemma 2. 1, $\{T_s x : s \in G\}$ is bounded. So, there exists a subnet $\{T_{s_\alpha} x\}$ of the net $\{T_s x : s \in G\}$ which converges weakly to some $y \in C$. Since $\omega_w(x) \subseteq F(\mathcal{T})$ and $y \in \bigcap_s \overline{\text{co}}\{T_t x : t \geqq s\}$, we have $y \in \bigcap_s \overline{\text{co}}\{T_t x : t \geqq s\} \cap F(\mathcal{T})$. Therefore, it follows from Proposition 2. 4 that $\{T_s x : s \in G\}$ converges weakly to $y \in F(\mathcal{T})$.

Let K be a subset of a Banach space X. A mapping $T: K \to K$ is called asymptotically nonexpansive [4] if for each $x, y \in K$,

$$\|T^k x - T^k y\| \leqq a_k \|x-y\|, \quad k=1, 2, \cdots,$$

where $\{a_k\}$ is a fixed sequence of real numbers such that $\lim_{k\to\infty} a_k = 1$. It is proved in [4] that if K is a bounded closed and convex subset of a uniformly convex space X then the set $F(T)$ of fixed points of T is nonempty closed and convex. Taking $G=N$ in Theorem 2. 5, we have

COROLLARY 2. 6. *Let C be a closed convex and bounded subset of a uniformly convex Banach space X with a Fréchet differentiable norm. Let $x \in C$. If $T: C \to C$ is asymptotically nonexpansive mapping and $\omega_w(x) \subseteq F(T)$, then the sequence $\{T^n x : n \in N\}$ converges weakly to a fixed point of T, where $\omega_w(x)$ denotes the set of subsequentinl limits of $\{T^n x\}$.*

3. Strong convergence and fixed point

Throughout this section, G denotes a commutative semitopological semigroup with the identity, directed by an order relation defined by $t \geqslant s$ if and only if $t=as$ for some $a \in G$. Let C be a closed covex subset of a uniformly convex Banach space X. We suppose that the semigroup $\mathcal{T} = \{T_s : s \in G\}$ is of asymptotically nonexpansive type and, for each $x \in C$, its orbit $\mathcal{T}(x) = \{T_s x : s \in G\}$ is bounded.

By slight modification of Theorem 1 in [11], we have the following :

THEOREM 3. 1. *For each $x \in C$, the asymptotic center $c(x)$ of the orbit $\mathscr{T}(x)$ with respect to C is a common fixed point of $\mathscr{T}$.*

LEMMA 3. 2. *For each $x \in C$, $\lim_s PT_s x$ exists, where P is the metric projection of X onto $F(\mathscr{T})$.*

Proof. Let $r_s = \|T_s x - PT_s x\|$. With a proof as in Lemma 2. 1, we have $r = \inf_s \|T_s x - PT_s x\| = \limsup_s \|T_s x - PT_s x\|$. If $r=0$, then $\{PT_s x : s \in G\}$ is cleary a Cauchy net. For $r>0$, suppose that $\{PT_s x\}$ is not a Cauchy net. Then, there exists $\varepsilon>0$ and $\{s_\alpha, t_\alpha\} \subseteq G$ such that

$$\|PT_{s_\alpha} x - PT_{t_\alpha} x\| \geqq \varepsilon$$

for every α. Now choose a $\sigma>0$ so small that

$$(r+\sigma)\left[1-\delta\left(\frac{\varepsilon}{r+\sigma}\right)\right] < r,$$

where δ is the modulus of convexity of the norm.

For the $\sigma>0$, there is $s_\alpha, t_\alpha \in G$ such that $r_{s_\alpha}, r_{t_\alpha} < r + \frac{\sigma}{2}$. Since $\mathscr{T}$ is of asymptotically nonexpansive type, there is $t_0 \in G$ such that

$$\|T_t T_{s_\alpha} x - PT_{s_\alpha} x\| \leqq \|T_{s_\alpha} x - PT_{s_\alpha} x\| + \frac{\sigma}{2} < r + \sigma$$

and also

$$\|T_t T_{t_\alpha} x - PT_{t_\alpha} x\| < r + \sigma,$$

for all $t \geqq t_0$ Taking $b = t_\alpha s_\alpha t_0$, by commutativity of G, we have that $\|T_b x - PT_{s_\alpha} x\|$, $\|T_b x - PT_{t_\alpha} x\| < r + \sigma$ and $\|PT_{s_\alpha} x - PT_{t_\alpha} x\| \geqq \varepsilon$. So, by uniform convexity of X, we have

$$\|T_b x - (PT_{s_\alpha} x + PT_{t_\alpha} x)/2\| \leqq (r+\sigma)\left[1-\delta\left(\frac{\varepsilon}{r+\sigma}\right)\right].$$

Thus,

$$\begin{aligned}\|T_b x - PT_b x\| &\leqq \|T_b x - (PT_{s_\alpha} x + PT_{t_\alpha} x)/2\| \\ &\leqq (r+\sigma)\left[1-\delta\left(\frac{\varepsilon}{r+\sigma}\right)\right] < r.\end{aligned}$$

This contradicts $r = \inf_s \|T_s x - PT_s x\|$. The proof is completed.

THEOREM 3. 3. *For each $x \in C$, $\lim PT_s x = c(x)$, where $c(x)$ is the asymptotic center of $\mathscr{T}(x)$ with respect to C.*

Proof. By Lemma 3. 2, there exists $z\in C$ such that $PT_sx \to z$. It suffices to show that $z=c(x)$. Indeed,

$$\begin{aligned}\lim_s \sup\|T_sx-z\| &\leqq \lim_s \sup\{\|T_sx-PT_sx\|+\|PT_sx-z\|\}\\ &\leqq \lim_s \sup\|T_sx-PT_sx\|\\ &\leqq \lim_s \sup\|T_sx-c(x)\|.\end{aligned}$$

Hence the uniqueness of asymptotic center implies that $z=c(x)$ (see[6]).

Remark. With a proof as in Theorem 3. 3, it is clear that if X satisfies Opial's condition ([15], [13 ; Lemma 2. 1]), and $\{T_sx : s\in G\}$ converges weakly to a $y\in F(\mathscr{T})$, then the net $\{PT_sx : s\in G\}$ converges strongly to the same fixed point y. It is easy that if C is a closed convex subset of a uniformiy convex space space X and if $\mathscr{T}$ is a right reversible semigroup of asymptotically nonexpansive type on C, then the set $F(\mathscr{T})$ of common fixed points is closed convex.

Finally, employing the method of the proof due to Goebel-Kirk-Thele [5 ; Theorem 3. 1]. For each $s\in G$ and $x\in C$, we denote by $\omega^s(x)$ the set of subnet limits of the net $\{T_{ts}x : t\in G\}$.

Lemma 3. 4. *Let C be a compact convex subset of a Banach space X and let G, $\mathscr{T}=\{T_s : s\in G\}$ be as before. Then there exists two subsets M and H of C satisfying the following properties:*

(a) $H\subseteq C$ is minimal with respect to being nonempty, closed, convex and satisfying that

(*) for each $x\in H$ and $s\in G$, $\omega^s(x)\subseteq H$;

(b) $M\subseteq H$ is minimal with respect to being nonempty, closed and satisfying that

(**) for each $x\in M$ and $s\in G$, $\omega^s(x)\subseteq M$;

(c) $M\subseteq \bigcap_{s\in G}\{T_s(M)\}$.

Proof. Use Zorn's lemma to obtain the subset H of C which is minimal with respect to being nonempty, closed, convex and satisfying the property (*). Again, we use Zorn's lemma to obtain the subset M of H which is minimal with respect to being nonempty, closed and satisfying the property (**). To prove (c), we note first that if $x\in M$

and $w \in \omega^t(x)$ for some $t \in G$, say, $\lim_\alpha T_{t_\alpha t} x = w$ for some subnet $\{t_\alpha\}$ of G, then $\lim_\alpha T_{st_\alpha t} x = T_s w \in M$ by (**). Therefore, for each $s \in G$,

$$H_{\mathscr{T}} = M \cap T_s(M) \neq \emptyset.$$

Obviously $T_s(M)$ is nonempty and closed. By minimality of M, to prove that $H_{\mathscr{T}} = M$, it suffices to show that for each $x \in H_{\mathscr{T}}$ and $t \in G$, $\omega^t(x) \subseteq H_{\mathscr{T}}$. Indeed, let $z \in \omega^t(x)$, say $z = \lim_\alpha T_{t_\alpha t} x$ for some subnet $\{t_\alpha\}$ of G. Then, since $x \in M$, $z \in M$ by (**). Also $x \in T_s(M)$ implies that $x = T_s y$ for some $y \in M$. By continuity and commutativity of members of $\mathscr{T}$, we get

$$\begin{aligned} \lim_\alpha T_{t_\alpha t} x &= \lim_\alpha T_{t_\alpha t} T_s y \\ &= T_s(\lim_\alpha T_{t_\alpha t}\, y) = T_s v. \end{aligned}$$

This implies $z = T_s v$ for some $v \in M$; hence $z \in M \cap T_s(M) = H_{\mathscr{T}}$. Thus, $T_s(M) \supseteq M$ and since $s \in G$ is arbitrary, (c) is proved.

THEOREM 3. 5. *Let C be a compact convex subset of a Banach space X and let G, $\mathscr{T}$ as before. Then $\mathscr{T}$ has a common fixed point in C.*

Proof. Let M, H be given as in Lemma 3. 4. Then it suffices to show that $\operatorname{diam}(M) = 0$. Now suppose that

$$\delta = \operatorname{diam}(M) > 0.$$

Since $\overline{\operatorname{co}}(M) = H$, by Lemma 1 in [1], there exists $r < \delta$ such that for some $u \in H$,

$$\sup\{\|u - x\| : x \in M\} \leqq r.$$

Set

$$D = \{x \in H : M \subseteq B(x, r)\}, \quad \text{where } B(x, r) = \{u \in X : \|u - x\| \subseteq r\}.$$

Since $u \in D$, D is nonempty, closed and convex subset of H. Moreover, because $\delta > r$ and D can not contain points of M whose distance exceeds r, it follows that D is a proper subset of H.

Next, let $z \in D$ and suppose $\lim_\alpha T_{t_\alpha s} z = w$ for some $s \in G$ and some subnet $\{t_\alpha\}$ of G. To show that $w \in D$, let $y \in M$. Since $T_{t_\alpha s}(M) \supseteq M$ for every $t_\alpha \in G$ by (c) of Lemma 3. 4, there exists $u_\alpha \in M$ such that $y = T_{t_\alpha s} u_\alpha$. Therefore, we have

$$\begin{aligned} \|w - y\| &\leqq \|w - T_{t_\alpha s} z\| + \|T_{t_\alpha s} z - y\| \\ &= \|w - T_{t_\alpha s} z\| + [\|T_{t_\alpha s} z - T_{t_\alpha s} u_\alpha\| \\ &\quad - \|z - u_\alpha\|] + \|z - u_\alpha\| \end{aligned}$$

Taking both sides by $\limsup_{\alpha}$, we obtain

$$\|w-y\| \leqq r.$$

Since $w \in H$ by (*), $w \in D$. This contradicts the minimality of H; hence $\delta=0$. The proof is completed.

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