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A NOTE ON GENERALIZED LIBERA OPERATOR

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1. Introduction.

Let A denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \tag{1.1}$$

which are analytic in the unit disk $U=\{z: |z|<1\}$. Let S be the class of all functions $f(z)$ in A which are univalent in the unit disk U.

A function $f(z)$ belonging to A is said to be starlike of order α if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \tag{1.2}$$

for some $\alpha(0\leqq\alpha<1)$, and for all $z\in U$. We denote by $S^*(\alpha)$ the subclass of A consisting of all functions $f(z)$ which are starlike of order α in the unit disk U. Throughout this paper, it should be understood that functions such as $zf'(z)/f(z)$, which have removable singularities at $z=0$, have had these singularities removed in statements like (1.2).

A function $f(z)$ belonging to A is said to be convex of order α if any only if

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \tag{1.3}$$

for some $\alpha(0\leqq\alpha<1)$, and for all $z\in U$. Also we denote by $K(\alpha)$ the subclass of A consisting of all functions $f(z)$ which are convex of order α in the unit disk U.

We note that $f(z)\in K(\alpha)$ if and only if $zf'(z)\in S^*(\alpha)$, and that $S^*(\alpha)\subseteq S^*(0)\equiv S^*$, $K(\alpha)\subseteq K(0)\equiv K$, and $K(\alpha)\subset S^*(\alpha)$ for $0\leqq\alpha<1$.

The classes $S^*(\alpha)$ and $K(\alpha)$ were first introduced by Robertson [1], and studied subsequently by Schild [2], MacGregor [3], Pinchuk [4], and others.

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Many essentially equivalent definitions of the fractional calculus (that is, fractional derivatives and fractional integrals) have been given in the literature ([5], [6], [7], and [8]). For our discussion, it is more convenient to use the following definitions which were employed recently by Owa ([9], and [10]).

DEFINITION 1. The fractional integral of order λ is defined, for a function $f(z)$, by

$$D_z^{-\lambda}f(z)=\frac{1}{\Gamma(\lambda)}\int_0^z\frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}d\zeta, \tag{1.4}$$

where $\lambda>0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

DEFINITION 2. The fractional derivative of order λ is defined, for a function $f(z)$, by

$$D_z^{\lambda}f(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z\frac{f(\zeta)}{(z-\zeta)^{\lambda}}d\zeta, \tag{1.5}$$

where $0\leqq\lambda<1$, $f(z)$ is an analytic in a simply connected region of the z-plane, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in Definition 1 above.

DEFINITION 3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+\lambda)$ is defined by

$$D_z^{n+\lambda}f(z)=\frac{d^n}{dz^n}D_z^{\lambda}f(z), \tag{1.6}$$

where $0\leqq\lambda<1$ and $n\in N_0=\{0,1,2,\cdots\}$.

Let $S^*(\alpha,\lambda)$ be the subclass of A consisting of functions $f(z)$ which satisfy

$$\mathrm{Re}\left\{\frac{\Lambda(\lambda,f)}{f(z)}\right\}>\alpha \tag{1.7}$$

for some $\lambda(\lambda<1)$, $\alpha(0\leqq\alpha<1)$, and for all $z\in U$, where

$$\Lambda(\lambda,f)=\Gamma(1-\lambda)z^{1+\lambda}D_z^{1+\lambda}f(z).$$

Also let $K(\alpha,\lambda)$ be the subclass of A consisting of functions $f(z)$ such that $\Lambda(\lambda,f)\in S^*(\alpha,\lambda)$. Since $S^*(\alpha,0)=S^*(\alpha)$ and $K(\alpha,0)=K(\alpha)$ when $\lambda=0$, $S^*(\alpha,\lambda)$ and $K(\alpha,\lambda)$ are the generalizations of the classes $S^*(\alpha)$ and $K(\alpha)$, respectively. The classes $S^*(\alpha,\lambda)$ and $K(\alpha,\lambda)$ were introduced by Owa [11]. Furthermore, recently, Owa and Shen [12] proved some coefficient inequalities for functions $f(z)$ belonging to $S^*(\alpha, \lambda)$ and $K(\alpha, \lambda)$.

Let T be the subclass of S whose members have the form

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n \qquad (a_n \leq 0). \tag{1.9}$$

We denote by $T^*(\alpha, \lambda)$ and $C(\alpha, \lambda)$ the classes obtained by taking intersections, respectively, of the classes $S^*(\alpha, \lambda)$ and $K(\alpha, \lambda)$ with T, that is,

$$T^*(\alpha, \lambda)=S^*(\alpha, \lambda) \cap T \tag{1.10}$$

and

$$C(\alpha, \lambda)=K(\alpha, \lambda) \cap T. \tag{1.11}$$

The classes $T^*(\alpha, \lambda)$ and $C(\alpha, \lambda)$ were studied by Owa [11], and the special cases $T^*(\alpha, 0)$ and $C(\alpha, 0)$ are the classes $T^*(\alpha)$ and $C(\alpha)$, respectively, by Silverman [13].

2. Starlikeness and convexity of the generalized Libera operator.

For a function $f(z)$ belonging to the class A, we define the generalized Libera operator $J_c(f)$ by

$$J_c(f)=\frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\,dt \qquad (c>-1). \tag{2.1}$$

The operator $J_c(f)$ when $c \in N=\{1, 2, 3, \ldots\}$ was studied by Bernardi [14]. In particular, the operator $J_1(f)$ was studied by Libera [15], Livingston [16], and Mocanu, Reade and Ripeanu [17]. More recently, Owa and Shen [18] proved the following results.

THEOREM A. *Let the function $f(z)$ defined by* (1.1) *be in the class $S^*(\alpha, \lambda) \cap S^*$ $(0 \leqq \alpha < 1;\ \lambda < 1)$. Then the operator $J_c(f)$ which $c \geqq 0$ is in the class $S^*(\alpha, \lambda)$.*

THEOREM B. *Let the function $f(z)$ defined by* (1.1) *be in the class $K(\alpha, \lambda)$ with α $(0 \leqq \alpha < 1)$ and λ $(\lambda < 1)$ such that $S^*(\alpha, \lambda) \subseteq S^*$. Then the operator $J_c(f)$ with $c \geqq 0$ is in the class $K(\alpha, \lambda)$.*

In order to derive our results, we have to recall here the following lemmas due to Owa [11].

LEMMA 1. *Let the function $f(z)$ be defined by* (1.9). *Then $f(z)$ belongs to the class $T^*(\alpha, \lambda)$ if and only if*

$$\sum_{n=2}^{\infty}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}a_n\leqq 1-\alpha \tag{2.2}$$

for $\lambda<1$ *and* $0\leqq\alpha<1$. *The result* (2.2) *is sharp.*

LEMMA 2. *Let the function* $f(z)$ *be defined by* (1.9). *Then* $f(z)$ *belongs to the class* $C(\alpha, \lambda)$ *if and only if*

$$\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}a_n\leqq 1-\alpha \tag{2.3}$$

for $\lambda<1$ *and* $0\leqq\alpha<1$. *The result* (2.3) *is sharp.*

Further, we need the following results by Silverman [13].

LEMMA 3. *Let the function* $f(z)$ *defined by* (1.1) *satisfy the inequality*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n|\leqq 1-\alpha \tag{2.4}$$

for $0\leqq\alpha<1$. *Then* $f(z)$ *is in the class* $S^*(\alpha)$.

LEMMA 4. *Let the function* $f(z)$ *defined by* (1.1) *satisfy the inequality*

$$\sum_{n=2}^{\infty}n(n-\alpha)|a_n|\leqq 1-\alpha \tag{2.5}$$

for $0\leqq\alpha<1$. *Then* $f(z)$ *is in the class* $K(\alpha)$.

Now, we state and prove

THEOREM 1. *Let the function* $f(z)$ *defined by* (1.1) *satisfy the inequality*

$$\sum_{n=2}^{\infty}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}|a_n|\leqq 1-\alpha \tag{2.6}$$

for $0<\lambda<1$ *and* $0\leqq\alpha<1$. *Then* $J_c(f)$ *defined by* (2.1) *is in the class* $S^*(\beta)$, *where*

$$\beta=1-\frac{(1-\alpha)(1-\lambda)(1+c)}{(2+c)\{2-\alpha(1-\lambda)\}-(1-\alpha)(1-\lambda)(1+c)}, \tag{2.7}$$

$\lambda c\geqq 1$, *and* $(1-2\lambda)c\leqq 1+4\lambda$. *The result is sharp.*

Proof. Note that

$$\begin{aligned} J_c(f)&=\frac{c+1}{z^c}\int_0^z t^{c-1}\left[t+\sum_{n=2}^{\infty}a_n t^n\right]dt\\ &=z+\sum_{n=2}^{\infty}\left[\frac{1+c}{n+c}\right]a_n z^n. \end{aligned} \tag{2.8}$$

In view of Lemma 3, we know that $J_c(f)\in S^*(\beta)$ if

$$\sum_{n=2}^{\infty}(n-\beta)\left[\frac{1+c}{n+c}\right]|a_n|\leqq 1-\beta, \tag{2.9}$$

or if

$$\sum_{n=2}^{\infty}\frac{(n-\beta)(1+c)}{(1-\beta)(n+c)}|a_n|\leqq 1 \qquad (n\geqq 2). \tag{2.10}$$

By virtue of (2.6), the inequality (2.10) holds true if

$$\frac{n-\beta}{1-\beta}\leqq\frac{(n+c)\left\{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}}{(1+c)(1-\alpha)} \qquad (n\geqq 2). \tag{2.11}$$

Hence, we only need to find the largest value of β such that

$$\beta\leqq\frac{(n+c)\left\{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}-n(1-\alpha)(1+c)}{(n+c)\left\{\dfrac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}-(1-\alpha)(1+c)} \tag{2.12}$$

for all $n\geqq 2$. Defining $H(n,\alpha,\lambda,c)$ by

$$H(n,\alpha,\lambda,c)=\frac{(n+c)G(n,\alpha,\lambda)-n(1-\alpha)(1+c)}{(n+c)G(n,\alpha,\lambda)-(1-\alpha)(1+c)} \tag{2.13}$$

with

$$G(n,\alpha,\lambda)=\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha, \tag{2.14}$$

we have that

$$H(n+1,\alpha,\lambda,c)-H(n,\alpha,\lambda,c)\geqq 0 \tag{2.15}$$

if

$$(n-1)(n+1+c)G(n+1,\alpha,\lambda)-n(n+c)G(n,\alpha,\lambda)+(1-\alpha)(1+c)\geqq 0 \tag{2.16}$$

for fixed α,λ,c. Furthermore, (2.16) holds true if

$$\{(1+\lambda)n^2+(\lambda c-1)n-(1+c)\}\Gamma(n+1)\Gamma(1-\lambda) + \alpha(1+c)\Gamma(n+1-\lambda)\geqq 0, \tag{2.17}$$

or

$$(1+\lambda)n^2+(\lambda c-1)n-(1+c)\geqq 0 \tag{2.18}$$

for all $n\geqq 2$. The inequality (2.18) is held provided that $0<\lambda<1$, $\lambda c\geqq 1$, and $(1-2\lambda)c\leqq 1+4\lambda$. This shows that $H(n,\alpha,\lambda,c)$ is an increasing function of n $(n\geqq 2)$ provided that $0\leqq\alpha<1$, $0<\lambda<1$, $\lambda c\geqq 1$, and $(1-2\lambda)c\leqq 1+4\lambda$. Consequently, we conclude that

$$\begin{aligned}\beta &\leqq \inf_{n\geqq 2} H(n, \alpha, \lambda, c)\\ &= H(2, \alpha, \lambda, c)\\ &= 1-\frac{(1-\alpha)(1-\lambda)(1+c)}{(2+c)\{2-\alpha(1-\lambda)\}-(1-\alpha)(1-\lambda)(1+c)}. \quad (2.19)\end{aligned}$$

Finally, by taking the function $f(z)$ given by

$$f(z)=z+\frac{(1-\alpha)(1-\lambda)}{2-\alpha(1-\lambda)}z^2 \quad (2.20)$$

for $0\leqq\alpha<1$ and $0<\lambda<1$, we have

$$J_c(f)=z+\frac{(1-\alpha)(1-\lambda)(1+c)}{(2+c)\{2-\alpha(1-\lambda)\}}z^2. \quad (2.21)$$

It follows from (2.21) that

$$\begin{aligned}\mathrm{Re}\left\{\frac{z(J_c(f))'}{J_c(f)}\right\} &= \mathrm{Re}\left\{\frac{1+\dfrac{2(1-\alpha)(1-\lambda)(1+c)}{(2+c)\{2-\alpha(1-\lambda)\}}z}{1+\dfrac{(1-\alpha)(1-\lambda)(1+c)}{(2+c)\{2-\alpha(1-\lambda)\}}z}\right\}\\ &= 1-\frac{(1-\alpha)(1-\lambda)(1+c)}{(2+c)\{2-\alpha(1-\lambda)\}-(1-\alpha)(1-\lambda)(1+c)} \quad (2.22)\end{aligned}$$

for $z=1$, which proves that the result of the theorem is sharp.

Corollary 1. *Let the function $f(z)$ defined by (1.9) be in the class $T^*(\alpha, \lambda)$ with $0<\lambda<1$ and $0\leqq\alpha<1$. Then $J_c(f)$ defined by (2.1) is in the class $T^*(\beta)$, where β is given by (2.7), $\lambda c\geqq 1$, and $(1-2\lambda)c\leqq 1+4\lambda$. The result is sharp.*

The proof of Corollary 1 follows from Lemma 1 and Theorem 1.

Letting $\alpha=0$ in Corollary 1, we have

Corollary 2. *Let the function $f(z)$ defined by (1.9) be in the class $T^*(0, \lambda)$ with $0<\lambda<1$. Then $J_c(f)$ defined by (2.1) is in the class $T^*(\beta)$, where*

$$\beta=\frac{2(1+\lambda+\lambda c)}{2+(1+\lambda)(1+c)}, \quad (2.23)$$

$\lambda c\geqq 1$, and $(1-2\lambda)c\leqq 1+4\lambda$. The result is sharp.

Taking $\lambda=1/2$ and $c=2$, Corollary 1 derives

Corollary 3. *Let the function $f(z)$ defined by (1.9) be in the class $T^*(\alpha, 1/2)$ with $0\leqq\alpha<1$. Then $J_2(f)$ defined by (2.1) when $c=2$ is* *in the class* $T^*((10+2\alpha)/(13-\alpha))$. *The result is sharp.*

Next, we prove

THEOREM 2. *Let the function* $f(z)$ *defined by* (1. 1) *satisfy the inequality*

$$\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}\left\{\frac{\Gamma(n+1)\Gamma(1-\lambda)}{\Gamma(n-\lambda)}-\alpha\right\}|a_n|\leqq 1-\alpha \tag{2.24}$$

for $\lambda<1$ *and* $0\leqq\alpha<1$. *Then* $J_c(f)$ *defined by* (2. 1) *is in the class* $K(\beta)$, *where* β *is given by* (2. 7), $\lambda c\geqq 1$, *and* $(1-2\lambda)c\leqq 1+4\lambda$. *The result is sharp.*

Proof. Note that $f(z)$ satisfies the inequality (2. 24) if and only if $\Lambda(\lambda,f)$ defined by (1. 8) satisfies (2. 6). Therefore, by using Theorem 1, we have $J_c(\Lambda(\lambda,f))\in S^*(\beta)$. This implies that

$$\begin{aligned} J_c(\Lambda,\lambda)\in S^*(\beta) &\Longrightarrow \Lambda(\lambda, J_c(f))\in S^*(\beta)\\ &\Longrightarrow J_c(f)\in K(\beta)\end{aligned}$$

provided that $0\leqq\alpha<1$, $\lambda<1$, $\lambda c\geqq 1$, and $(1-2\lambda)c\leqq 1+4\lambda$. Further, by considering the function $f(z)$ given by

$$f(z)=z+\frac{(1-\alpha)(1-\lambda)^2}{2\{2-\alpha(1-\lambda)\}}z^2 \tag{2.25}$$

for $0\leqq\alpha<1$ and $\lambda<1$, we can show that the result of the theorem is sharp.

Applying Lemma 2 and Theorem 2, we have

COROLLARY 4. *Let the function* $f(z)$ *defined by* (1. 9) *be in the class* $C(\alpha,\lambda)$ *with* $\lambda<1$ *and* $0\leqq\alpha<1$. *Then* $J_c(f)$ *defined by* (2. 1) *is in the class* $C(\beta)$, *where* β *is given by* (2. 7), $\lambda c\geqq 1$, *and* $(1-2\lambda)c\leqq 1+4\lambda$. *The result is sharp.*

Making $\alpha=0$, Corollary 4 gives

COROLLARY 5. *Let the funcion* $f(z)$ *defined by* (1. 9) *be in the class* $C(0,\lambda)$ *with* $\lambda<1$. *Then* $J_c(f)$ *defined by* (2. 1) *is in the class* $C(\beta)$, *where* β *is given by* (2. 23), $\lambda c\geqq 1$, *and* $(1-2\lambda)c\leqq 1+4\lambda$. *The result is sharp.*

Finally, taking $\lambda=1/2$ and $c=2$ in Corollary 4, we have

COROLLARY 6. *Let the function* $f(z)$ *defined by* (1. 9) *be in the class* $C(\alpha,1/2)$ *with* $0\leqq\alpha<1$. *Then* $J_2(f)$ *defined by* (2. 1) *when* $c=2$ *is* *in the class* $C((10+2\alpha)/(13-\alpha))$. *The result is sharp.*

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