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FUNCTIONS IN THE BANACH AIGEBRA $S(v)$

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1. Introduction and Preliminaries.

Let ν be a positive integer and let $C_0^\nu[0,t]$ denote ν-dimensional Wiener space, that is the space of $\boldsymbol{R}^\nu$-valued continuous functions $\vec{X}=(x_1,\cdots,x_\nu)$ on $[0,t]$ such that $\vec{X}(0)=\vec{0}$. In case $\nu=1$ we suppress the ν and simply write $x\in C_0[0,t]$. In [2] Cameron and Storvick introduced a Banach algebra $S(\nu)$ of (equivalence classes of) functions on Wiener space which are a type of stochastic Fourier transform of finite Borel measures. They showed [2, Theorem 5. 1] that the analytic Feynman integral exists for all elements of $S(\nu)$. In this paper we establish a general theorem insuring that various functions $f: C_0^\nu[0,t]\to \boldsymbol{C}$ belong to $S(\nu)$. We then give several corollaries which contain, with the exception of some results in [6, 16, 19] on quadratic potentials, all of the results that we know of to date which insure that various functions of interest in connection with the Feynman integral and quantum mechanics are in $S(\nu)$ for some ν. The results of this paper combined with the results in [6, 16, 19] show that $S(\nu)$ contains a broad class of functions.

Next we give the definition of $S(\nu)$. Let m^ν denote ν-dimensional Wiener measure. A subset A of $C_0^\nu[0,t]$ is said to be scale-invariant measurable [14] provided ρA is Wiener measurable for every $\rho>0$, and a scale invariant measurable set N is said to be scale-invariant null provided $m^\nu(\rho N)=0$ for every $\rho>0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a. e.). Let $M(L_2^\nu[0,t])$ denote the collection of complex-valued countably additive measures on $\beta(L_2^\nu[0,t])$, the Borel class of $L_2^\nu[0,t]$.

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$M(L_2^\nu)$ is a Banach algebra under the total variation norm where convolution is taken as the multiplication.

Let σ be in $M(L_2^\nu)$. Consider the function $\tilde{\sigma}$ defined for s-a. e. $\vec{X}=(x_1, \cdots, x_\nu)$ in $C_2^\nu[0,t]$ by the formula

$$(1.1) \qquad \tilde{\sigma}(\vec{X})=\int_{L_2^\nu[0,t]} \exp\left\{i\sum_{j=1}^{\nu}\int_0^t v_j(s)\tilde{d}x_j(s)\right\}d\sigma(\vec{V})$$

where $\vec{V}=(v_1, \cdots, v_n)$ and where $\int_0^t v_j(s)\tilde{d}x_j(s)$ denotes the Paley-Wiener-Zygmund stochastic integral [6, or 16]. The elements of $S(\nu)$ consist of equivalence classes $[\tilde{\sigma}]$ of functions which are s-a. e. equal to $\tilde{\sigma}$ for some σ in $M(L_2^\nu[0,t])$. One often uses loose terminology and refers to the elements of $S(\nu)$ as functions. Cameron and Storvick [2] show that the correspondence $\sigma\to[\tilde{\sigma}]$ is injective, carries convolution into pointwise multiplication and that $S(\nu)$ is a Banach algebra with norm $||[\tilde{\sigma}]||=||\sigma||$.

2. The General Theorem.

In this section we establish the theorem discussed in the introduction.

THEOREM 1. *Let λ be a finite Borel measure on $[0,t]^n$. For $j=1,2,\cdots,n$ let $\phi_j:[0,t]\to L_2[0,t]$ be Borel measurable. Let $\theta:[0,t]^n\times \boldsymbol{R}^{n\nu}\to\boldsymbol{C}$ be such that for all $\vec{r}=(r_1,\cdots,r_n)\in[0,t]^n$,*

$$(2.1) \qquad \theta(\vec{r}\,;\vec{U}_1,\cdots,\vec{U}_n)=\int_{\boldsymbol{R}^{n\nu}} exp\left\{i\sum_{j=1}^{n}\langle\vec{U}_j,\vec{V}_j\rangle\right. d\sigma_{\vec{r}}(\vec{V}_1,\cdots,\vec{V}_n) =\hat{\sigma}_{\vec{r}}((\vec{U}_1,\cdots,\vec{U}_n)$$

where $\sigma_{\vec{r}}\in M(\boldsymbol{R}^{n\nu})$, the measure algebra of $\boldsymbol{R}^{n\nu}$, $\vec{U}_j=(u_{j1},\cdots,u_{j\nu})\in\boldsymbol{R}^\nu$,

(2.2) for every $B\in\beta(\boldsymbol{R}^{n\nu})$, $\sigma_{\vec{r}}(B)$ is a Borel measurable function of $\vec{r}$, and

$$(2.3) \qquad ||\sigma_{\vec{r}}||\in L_1([0,t]^n,\ \beta([0,t]^n),\lambda).$$

Then the function $f:C_0^\nu[0,t]\to\boldsymbol{C}$ defined by the formula

$$(2.4) \qquad f(\vec{X})=\int_{[0,t]^n}\theta\left(\vec{r}\,;\left(\int_0^t\phi_1(r_1)(s)\tilde{d}x_j(s)\right)_{j=1}^{\nu},\cdots,\left(\int_0^t\phi_n(r_n)(s)\tilde{d}x_j(s)\right)_{j=1}^{\nu}\right)d\lambda(\vec{r})$$

belongs to the Banach algebra $S(\nu)$.

Proof. To show that f is in $S(\nu)$ we need to find a measure σ in

$M(L_2^v[0,t])$ such that $\tilde{\sigma}(\vec{X})=f(\vec{X})$ for s-a. e. $\vec{X}=(x_1, \cdots, x_v)$ in $C_0^v[0,t]$ where $\tilde{\sigma}$ is given by (1. 1).

First using [13, Proposition 2] it is quite easy to see that θ is a Borel measurable function of $(\vec{r}\ ; \vec{U}_1, \cdots, \vec{U}_n)$. It then follows that the integrand on the right hand side of (2. 4) is a Borel measurable function of $\vec{r}$ since it is composed of Borel measurable functions.

Next we use the Borel measure λ and the family of Borel measures $\{\sigma_{\vec{r}} : \vec{r}\in[0,t]^n\}$ to construct a Borel measure μ on $[0,t]^n\times\boldsymbol{R}^{nv}$ by letting

$$\mu(E)=\int_{[0,t]^n}\sigma_{\vec{r}}(E^{(\vec{r})})d\lambda(\vec{r}) \tag{2. 5}$$

for $E\in\beta([0,t]^n\times\boldsymbol{R}^{nv})$. Using the Unsymmetric Fubini Theorem [13] we see that μ is an element of $M([0,t]^n\times\boldsymbol{R}^{nv})$.

Now for $j=1,2,\cdots,v$ let $\Phi_j : [0,t]^n\times\boldsymbol{R}^{nv}\to L_2[0,t]$ be defined by the formula

$$\Phi_j(s)\equiv\Phi_j(\vec{r}\ ; \vec{V}_1, \cdots, \vec{V}_n)(s)=\sum_{k=1}^{n}v_{kj}\phi_k(r_k)(s) \tag{2. 6}$$

where $\vec{V}_m=(v_{m1}, \cdots, v_{mv})\in\boldsymbol{R}^v$ for m$=1,\cdots,n$. Le t $\Phi : [0,t]^n\times\boldsymbol{R}^{nv}\to L_2^v[0,t]$ be defined by the formula

$$\Phi(s)=\Phi(\vec{r} : \vec{V}_1, \cdots, \vec{V}_n)(s)=(\Phi_1(s), \cdots, \Phi_v(s)).$$

Clearly Φ is Borel measurable.

Finally define σ in $M(L_2^v[0,t])$ by $\sigma\equiv\mu\circ\Phi^{-1}$. We need to show that $\tilde{\sigma}(\vec{X})=f(\vec{X})$ for s-a. e. $\vec{X}\in C_0^v[0,t]$. That is to say, for fixed $\rho>0$, we need to show that $\tilde{\sigma}(\rho\vec{X})=f(\rho\vec{X})$ for a. e. $\vec{X}\in C_0^v[0,t]$. But using the Change of Variables Theorem [11, p. 163] and the Unsymmetric Fubini Theorem [13] it follows that for a. e. $\vec{X}$ in $C_0^v[0,t]$,

$$\begin{aligned}\tilde{\sigma}(\rho\vec{X})&=\int_{L_2^v[0,t]}\exp\left\{i\rho\sum_{j=1}^{v}\int_0^t u_j(s)\tilde{d}x_j(s)\right\}d\sigma(u_1, \cdots, u_v)\\&=\int_{L_2^v[0,t]}\exp\left\{i\rho\sum_{j=1}^{v}u_j(s)\tilde{d}x_j(s)\right\}d(\mu\circ\Phi^{-1})(u_1, \cdots, u_v)\\&=\int_{[0,t]^n\times\boldsymbol{R}^{nv}}\exp\left\{i\rho\sum_{j=1}^{v}\int_0^t\Phi_j(s)\tilde{d}x_j(s)\right\}d\mu(\vec{r}\ ; \vec{V}_1, \cdots, \vec{V}_n)\\&=\int_{[0,t]^n}\left[\int_{\boldsymbol{R}^{nv}}\exp\left\{i\rho\sum_{k=1}^{n}\sum_{j=1}^{v}v_{kj}\int_0^t\phi_k(r_k)(s)\tilde{d}x_j(s)\right\}\right.\\&\qquad\qquad\left.d\sigma_{\vec{r}}(\vec{V}_1, \cdots, \vec{V}_n)\right]d\lambda(\vec{r})\\&=\int_{[0,t]^n}\left[\int_{\boldsymbol{R}^{nv}}\exp\left\{i\rho\sum_{k=1}^{n}\left\langle\left(\int_0^t\phi_k(r_k)(s)\tilde{d}x_j(s)\right)_{j=1}^{v}, (v_{k1}, \cdots, v_{kv})\right\rangle\right\}\right.\end{aligned}$$

$$d\sigma_{\vec{r}}(\vec{V}_1, \cdots, \vec{V}_n)\Big]d\lambda(\vec{r})$$

$$=\int_{[0,t]^n}\theta\Big(\vec{r}\ ;\ \Big(\rho\int_0^t\phi_1(r_1)(s)\tilde{d}x_j(s)\Big)_{j=1}^{\nu}, \cdots, \Big(\rho\int_0^t\phi_n(r_n)(s)\tilde{d}x_j(s)\Big)_{j=1}^{\nu}\Big)d\lambda(\vec{r})$$

$$=f(\rho\vec{X}).$$

Thus f is in $S(\upsilon)$ which completes the proof of Theorem 1.

We remark that we could have given an alternative proof of Theorem 1 using some recent results [7] concerning the Fresnel class $\mathcal{F}(H)$ of Fourier-Stieltjes transforms of bounded Borel measures on a separable infinite-dimensional Hilbert space H and the fact that for proper choice of H, $\mathcal{F}(H)$ and $S(1)$ are isometrically isomorphic [12]. However, under our present assumptions and state of knowledge, the proof involves certain measure theoretic technicalities which makes it more complicated than the proof that we have given above.

3. Various Corollaries of Theorem 1.

Our first corollary is an easy consequence of the fact that $S(\upsilon)$ is a Banach algebra. This result is relevant to quantum mechanics where exponential functions play a prominent role.

Corollary 1. *Let f be as in Theorem 1 and let h be an entire function on $\boldsymbol{C}$. Then $h(f(\vec{X}))$ is in $S(\upsilon)$. In particular exp $\{f(\vec{X})\}$ is in $S(v)$.*

Corollary 2. *Let $\theta : [0, t]^n \times \boldsymbol{R}^{n\nu} \to \boldsymbol{C}$ be given by (2. 1). Let $(s_1, s_2, \cdots, s_n)$ be any fixed point in $[0, t]^n$. Let*

(3. 1) $$f_1(\vec{X}) = \theta(s_1, \cdots, s_n\ ;\ \vec{X}(s_1), \cdots, \vec{X}(s_n)).$$

Then f_1 belongs to $S(\upsilon)$.

Proof. Apply Theorem 1 with $\phi_j(r_j)(s) = \chi_{[0,r_j]}(s)$ for $j=1, \cdots, n$, and with λ having unit mass concentrated at the point $(s_1, \cdots, s_n)$ (or let λ be any probability measure on $[0, t]^n$ and choose $\phi_j(r_j)(s) = \chi_{[0,s_j]}(s)$ for $j=1, 2, \cdots, n$).

Remark 1. Various results in [17], for example Proposition 3. 1, and Corollaries 3. 3 and 3. 5, now follow easily from Corollary 2 above by letting $n=1$ and $s_1=t$. The resulting functions arise in connection with the Schroedinger equation with potential $\theta=\theta(t, \vec{U})$.

REMARK 2. Let $\psi : \boldsymbol{R} \to \boldsymbol{C}$ and define $f_2 : C_0[0, t] \to \boldsymbol{C}$ by the formula

$$f_2(x) = \psi(x(t)). \tag{3.2}$$

Functions of the form (3.2) are of interest in quantum mechanics where ψ is the initial state of the quantum system. By Corollary 2 above we see that if $\psi = \hat{\sigma}$ with $\sigma \in M(\boldsymbol{R})$ then f_2 is in $S(1)$. It is interesting to ask if conditions insuring that a function is in $S(1)$ are necessary as well as sufficient. For a certain class of functions, including f_2, the authors [8] have recently shown that the answer is *yes*. In particular, if f_2 is given by (3.2), then f_2 is in $S(1)$ if and only if $\psi = \hat{\sigma}$ with $\sigma \in M(\boldsymbol{R})$.

COROLLARY 3. *Let λ be a finite Borel measure on $[0, t]^n$ and let*

$$f_3(\vec{X}) = \int_{[0,t]^n} \theta(r_1, \cdots, r_n\,;\, \vec{X}(r_1), \cdots, \vec{X}(r_n))\, d\lambda(r_1, \cdots, r_n) \tag{3.3}$$

where θ is given by (2.1). *Then f_3 belongs to $S(\upsilon)$.*

Proof. Apply Theorem 1 with $\phi_j(r_j)(s) = \chi_{[0, r_j]}(s)$ for $j = 1, \cdots, n$.

REMARK 3. Note that Corollary 3 above contains the main result of [15]. namely Theorem 1 on page 319. For simply choose $n=1$, $\upsilon=1$, and λ to be Lebesgue measure on $[0, t]$. Then

$$f_4(x) = \int_0^t \theta(r\,;\, x(r))\, dr \tag{3.4}$$

and $\exp\{f_4(x)\}$ are both clearly in $S(1)$.

In a recent expository essay [18], Nelson calls attention to some functions on Wiener space which were discussed in the book of Feynman and Hibbs [10, section 3–13] and in Feynman's original paper [9, section 13]. These functions have the form

$$f_5(x) = \exp\left\{\int_0^t \int_0^t \mathrm{W}(r_1, r_2\,;\, x(r_1), x(r_2))\, dr_1 dr_2.\right\}$$

Feynman obtained such functions by integrating out the oscillator coordinates in a system involving a harmonic oscillator interacting with a particle moving in a potential. Further functions like (3.5) but involving multiple integrals of more dimensions than two arise when more parti cles are involved. Our next corollary involves such functions.

COROLLARY 4. *Let λ be a finite Borel measure on $[0, t]^n$ and let*

$$f_6(\vec{X}) = exp\int_0^t \cdots \int_0^t \theta(r_1, \cdots, r_n\,;\, \vec{X}(r_1), \cdots, \vec{X}(r_n))\, d\lambda(r_1, \cdots r_n)\Big\} \tag{3.6}$$

with $\theta : [0,t]^n \times \boldsymbol{R}^{n\nu} \to \boldsymbol{C}$ *given by* (2.1). *Then* f_6 *belongs to* $S(\nu)$

REMARK 4. When Colollary 4 above is combined with the main results of [19, Theorems 3.1 and 4.1] involving quardratic potentials depending on n time parameters, one sees that a rather large class of functions involving n time parameters belong to $S(\nu)$ [19, Corollary 3.4].

In our next corollary we see that a Riemann sum approximation, $R_m(\vec{X})$, for $\int_0^t \cdots \int_0^t \theta(r_1, \cdots, r_n ; \vec{X}(r_1), \cdots, X(r_n))dr_1, \cdots, dr_n$ is in $S(\nu)$.

COROLLARY 5. *Let* $\theta : [0,t]^n \times \boldsymbol{R}^{n\nu} \to \boldsymbol{C}$ *be given by* (2.1) *and let*
$R_m(\vec{X}) = \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \cdots \sum_{j_n=1}^{m} (t/m)^n \, \theta\left(\frac{j_1 t}{m}, \frac{j_2 t}{m}, \cdots, \frac{j_n t}{m} ; \vec{X}\left(\frac{j_1 t}{m}\right), \cdots, \vec{X}\ \frac{j_n t}{m}\right.$
for $m=1,2,\cdots$. *Then* R_m *is in* $S(\nu)$ *for* $m=1,2,\cdots$.

Proof. For each $m=1, 2, \cdots$, apply Theorem 1 with $\phi_j(r_j)(s) = \chi_{[0,r_j]}(s)$ and with λ a discrete measure with mass $(t/m)^n$ at each of the m^n points $\left(\frac{j_1 t}{m}, \frac{j_2 t}{m}, \cdots, \frac{j_n t}{m}\right)$ in $(0,t]^n$.

Next we will show that functions of the form (3.1) (and (3.3)), but involving independent increments $\vec{X}(s_1)$, $\vec{X}(s_2)-\vec{X}(s_1)$, $\cdots$, $\vec{X}(s_n) -\vec{X}(s_{n-1})$ are in $S(\nu)$. To do this we first need to adjust equation (2.4) accordingly.

COROLLARY 6. *Let* $\phi_1, \cdots, \phi_n$, λ *and* θ *be as in Theorem* 1. *Then the function* $f_7 : C_0^\nu[0,t] \to \boldsymbol{C}$ *defined by the formula*

$$f_7(\vec{X}) = \int_{[0,t]^n} \theta\Big(r ; \Big(\int_0^t \phi_1(r_1)(s)\tilde{d}x_j(s)\Big)_{j=1}^{\nu}, \Big(\int_0^t [\phi_2(r_2)(s) -\phi_1(r_1)(s)]\tilde{d}x_j(s)\Big)_{j=1}^{\nu}, \cdots \Big(\int_0^t [\phi_n(r_n)(s)-\phi_{n-1}(r_{n-1})(s)]\tilde{d}x_j(s)\Big)_{j=1}^{\nu}\Big) d\lambda(\vec{r})$$

is in the Banach algebra $S(\nu)$.

Proof. The proof of this corollary parallels the proof of Theorem 1 above provided we replace equation (2.6) with

$$(2.6)' \quad \Phi_j(s) \equiv \Phi_j(\vec{r}, \vec{V}_1, \cdots, \vec{V}_n) = \sum_{k=1}^{n} v_{kj} \, [\phi_k(r_k)(s) - \phi_{k-1}(r_{k-1})(s)]$$

where $\phi_0(r_0)(s)$ is identically zero.

COROLLARY 7. *Let* θ *and* λ *be as in Theorem* 1 *and let*

$$f_8(\vec{X}) = \int_{[0,t]^n} \theta(\vec{r}, \vec{X}(r_1), \vec{X}(r_2)-\vec{X}(r_1), \cdots, \vec{X}(r_n)-\vec{X}(r_{n-1}))\, d\lambda(\vec{r})$$

Then f_8 *belongs to* $S(\nu)$.

Proof. In Corollary 6 choose $\phi_j(r_j)(s)=\chi_{[0,r_j]}(s)$ for $j=1, \cdots, n$.

COROLLARY 8. *Let* $0<s_1<s_2<\cdots<s_n\leq t$ *be a fixed partition of* $[0,t]$. *Let* $\vec{s}=(s_1, \cdots, s_n)$ *and let*

$$f_9(\vec{X})=\theta(\vec{s}\vec{X}(s_1), \vec{X}(s_2)-\vec{X}(s_1), \cdots, \vec{X}(s_n)-\vec{X}(s_{n-1}))$$

where θ *is given by* (2.1). *Then* f_9 *is in* $S(v)$.

Proof. In Corollary 7 let λ have unit mass at the point $\vec{s}=(s_1, \cdots, s_n)\in[0,t]^n$.

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