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SEMISIMPLE MALCEV-ADMISSIBLE MUTATION ALGEBRAS

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1. Introduction

In this paper we investigate the relationships between an alternative algebra A and its (p, q)-mutation algebra $A(p, q)$ in terms of simplicity and semisimplicity. The present discussion is a continuation of an earlier work by Myung and Shin [4] which concerns with nonassociative identities satisfied by $A(p, q)$.

We first recall some basic facts from [4]. Let B denote an (nonassociative) algebra over a field F with multiplication xy. The commutator $[x, y]$, anticommutator $\{x, y\}$ and associator (x, y, z) in B are defined by $[x, y]=xy-yx$, $\{x, y\}=xy+yx$ and $(x, y, z)=(xy)z-x(yz)$. For a multiplication denoted by $x*y$, these will be expressed by $[x, y]^*$, $\{x, y\}^*$ and $(x, y, z)^*$, respectively. Also, the commutative center $K(B)$, nucleus $N(B)$ and center $Z(B)$ of B are defined by $K(B)=\{x\in B \mid [x, B]=0\}$, $N(B)=\{x\in B \mid (x, B, B)=(B, x, B)=(B, B, x)=0\}$ and $Z(B)=K(B)\cup N(B)$. Thus, if $x\in N(B)$ then we can write xyz for $(xy)z=x(yz)$ for all $x, y, z\in B$. Attached to B are the anticommutative algebra B^- and the commutative algebra B^+ with multiplications $[x, y]$ and $\{x, y\}$ defined on the vector space B. The algebra B is called *Malcev-admissible* if B^- is a Malcev algebra, that is, the product $[x, y]$ satisfies the Malcev identity

(1) $[[x, y], [x, z]]=[[[x, y], z], x]+[[[y, z], x], x]+[[[z, x], x], y]$

for all $x, y, z\in B$, and B is called *Lie-admissible* if B^- is a Lie algebra, i.e., B^- satisfies the Jacobi identity $[[x, y], z]+[[y, z], x]+[[z, x], y]=0$. It is well known that any Lie-admissible algebra is Malcev-admissible and that an octonion algebra is Malcev-admissible but not

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Lie-admissible (see Myung [3]).

An algebra A over F is termed *alternative* if it satisfies the alternative laws $x^2y=x(xy)$ and $yx^2=(yx)x$ for all $x, y \in A$. Thus, any associative algebra is alternative, and it is well known that an octonion algebra is alternative but not associative [3]. Following Santilli's introduction of the (p, q)-mutation of an associative algebra, Myung [2] introduced the (left) *(p, q)-mutation* $A(p, q)$ of an alternative algebra A as the algebra with multiplication $x*y=(xp)y-(yq)x$ defined on the vector space A, where p and q are fixed elements in A. If p and q are in the nucleus $N(A)$, then $x*y$ is described by

$$x*y=xpy-yqx. \tag{2}$$

As note in [2], $A(p, q)$ is not in general alternative but is Malcev-admissible, that is, identity (1) holds for $A(p, q)$ with product $[x, y]^* =x*y-y*x$. As shown in [4], when p and q are in $N(A)$, the existence of a unit element e (i.e., $x*e=e*x=x$ for all $x \in A$) in $A(p, q)$ implies many of the well known identities for nonassociative algebras. The following result proved in [4] is instrumental for our investigation.

THEOREM 1. *Let A be an alternative algebra over a field F and let p, q be elements in the nucleus $N(A)$ of A such that $A(p, q)$ has a unit element e. Then*

(i) *A has a unit element 1.*

(ii) *$p-q$ is invertible in A and $e=(p-q)^{-1}$.*

(iii) *$s=(p-q)^{-1}q$ is an element in the center $Z(A)$.*

(iv) *The $(s+1, s)$-mutation $A(s+1, s)$ is isomorphic to $A(p, q)$ by the map*

$$f(x)=x(p-q)^{-1}, \quad x \in A. \tag{3}$$

(v) *$A(p, q)$ is power-associative, that is, every element in $A(p, q)$ generates an associative subalgebra.*

2. Simplicity in $A(p, q)$

By a theorem of Jacobson [5], the center $Z(B)$ of any simple algebra B over a field F is either zero or a field. In the latter case, B has a unit element e. Recall that an algebra B over F is called central simple over F if the scalar extension of B to any extension field K of F is simple over K. If B has a unit element e, then B is central simple over F if and only if B is simple and $Z(B)=Fe$ [5].

In the remainder of this paper, A will denote an alternative algebra over a field F of characteristic $\neq 2, 3$ with multiplication denoted by xy and p, q are fixed elements in the nucleus $N(A)$, so that the multiplication $x*y$ in the (p, q)-mutation $A(p, q)$ is given by relation (2). We begin with

THEOREM 2. *Assume that $A(p, q)$ has a unit element e. Then, A is simple over F if and only if $A(p, q)$ is simple over F.*

Proof. It is clear that if $A(p, q)$ is simple, then so is A, since an ideal of A is also an ideal of $A(p, q)$ by (2). Conversely, assume that A is simple. Denote by $A(p-q)$ the algebra with multiplication $x\cdot y =x(p-q)y$ defined on the vector space A. Since $p-q$ is invertible in A by Theorem 1, we can consider the map f given by (3) which is easily shown to be an isomorphism of A to $A(p-q)$. Thus, $A(p-q)$ is an alternative algebra also. Since $\{x, y\}^* = x(p-q)y+y(p-q)x$ by (2), we have the isomorphisms $A(p, q)^+ \simeq A(p-q)^+ \simeq A^+$. It follows from [6] that A is simple if and only if A^+ is simple. Hence, $A(p, q)^+$ is simple and so is $A(p, q)$.

For the relationship between $Z(A)$ and $Z(A(p, q))$, and for central simplicity in $A(p, q)$, we can show

THEOREM 3. *Assume that $A(p, q)$ has a unit element e. Then*

(i) $K(A(p, q)) = \{x \in A \mid (p+q)[x(p-q), A]=0\}$, *and if $p+q$ is not a zero divisor in A, then $K(A(p, q))=Z(A)e$.*

(ii) $Z(A)e \subset Z(A(p, q))$, *and if $p+q$ is not a zero divisor in A then $Z(A)e=Z(A(p, q))=K(A(p, q))$.*

In addition, let $p+q\neq 0$. Then,

(iii) *if A is simple then $Z(A)e=Z(A(p, q))=K(A(p, q))$.*

(iv) *A is central simple over F if and only if $A(p, q)$ is central simple over F.*

Proof. Note from Theorem 1 that the element $s=(p-q)^{-1}q$ is in the center $Z(A)$ and $A(s+1, s)$ is isomorphic to $A(p, q)$ under the map f defined by (3). Denote the multiplication, commutator and associator in $A(s+1, s)$ by $x\circ y$, $[x, y]^\circ = x\circ y - y\circ x$ and $(x, y, z)^\circ = (x\circ y)\circ z - x\circ(y\circ z)$. Using (2) with p and q replaced by $s+1$ and s, we have

(4) $[x, y]^{\circ} = (2s+1)[x, y]$,

(5) $(x, y, z)^{\circ} = (s+1)^2(x, y, z) - s^2(z, y, x) + s(s+1)[x(zy) + (yz)x - (yx)z - z(xy)]$.

We also note that

(6) $2s+1 = (p-q)^{-1}(p+q)$.

If $x \in K(A(p, q))$, then $[x, A]^* = 0$ and so $f^{-1}([x, A]^*) = [f^{-1}(x), A]^{\circ} = (2s+1)[x(p-q), A] = 0$ by (4), hence by (6) $(p+q)[x(p-q), A] = 0$. Similarly, the converse follows and this proves the first part of (i). We note from [6] that $3K(A) \subset N(A)$ for any alternative algebra A and hence $Z(A) = K(A) \cap N(A) = K(A)$, since the characteristic of F is not three. Let $xe \in Z(A)e$ for $x \in Z(A)$. Then, $f^{-1}([xe, A]^*) = [f^{-1}(xe), A]^{\circ} = [x, A]^{\circ} = (2s+1)[x, A] = 0$ by (6), and so $[xe, A]^* = 0$ to show that $Z(A)e \subset K(A(p, q))$. Since $x \in Z(A)$, from (5) we have $(x, y, z)^{\circ} = s(s+1)[[x, z], y] = 0$ for all $y, z \in A$ and similarly $(A, x, A)^{\circ} = (A, A, x)^{\circ} = 0$. Thus, $f^{-1}(x) = xe$ is in $N(A(p, q))$ for all $x \in Z(A)$, and this proves the first part of (ii). Assume now that $p+q$ is not a zero divisor in A. For $x \in K(A(p, q))$, we have $f^{-1}(x) = x(p-q)$ in $K(A(s+1, s))$ and hence by (4), (6) $[x(p-q), A] = 0$, so $x(p-q) \in Z(A)$ to show that $Z(A)e = K(A(p, q))$. In view of part (i), it suffices to show that $K(A(p, q)) \subset N(A(p, q))$. Thus, for $x \in K(A(p, q))$, we have $f^{-1}(x) = x(p-q)$ in $K(A(s+1, s))$ and by (4) $x(p-q) \in K(A) = Z(A)$. As before, by (5) this implies that $x(p-q)$ is in $N(A(s+1, s))$ and hence $f(x(p-q)) = x \in N(A(p, q))$, to show that $K(A(p, q)) \subset N(A(p, q))$ which proves part (ii).

Assume that $p+q \neq 0$. If A is simple, then $Z(A)$ is a field and $2s+1 = (p-q)^{-1}(p+q) \in Z(A)$ is invertible, so is $p+q$. Hence, part (iii) follows from (ii). If A is central simple over F, then $Z(A) = F1$ and by part (iii) $Z(A(p, q)) = Fe$, which shows that $A(p, q)$ is central simple over F, since $A(p, q)$ is simple over F by Theorem 1. Conversely, if $A(p, q)$ is central simple, then $Z(A)e \subset Z(A(p, q)) = Fe$, hence $Z(A) = F1$ and A is central simple over F.

Theorem 3 has been proved in [1] for the associative case. When A is a finite-dimensional simple alternative algebra over F, one can prove Theorem 3 without restrictions on $p+q$.

THEOREM 4. *Let* $K_2(A) = \{x \in A \mid [[x, A], A] = 0\}$. *Suppose that* A *is finite-dimensional simple over* F *and that* $A(p, q)$ *has a unit element* e.

Then

(i) $Z(A)e = Z(A(p, q)) = K_2(A)e$.

(ii) *A is central simple over F if and only if* $A(p, q)$ *is central simple over F.*

Proof. Since A is simple, $Z(A)$ is a field containing $F1$. Let $K = Z(A)$. Then, the scalar extension $K \otimes_F A = A_K$ of A to K is central simple over K[5]. Since $Z(A_K) = Z(A)_K$ and $K_2(A_K) = K_2(A)_K$, to show that $Z(A) = K(A)$ it suffices to assume that A is central simple over F. By the same argument, we can further assume that F is algebraically closed. We first contend that K_2(A) is an ideal of A^- for any alternative algebra A. For this, recall that a linearized form of Malcev identity (1) is given by $[[x, z], [y, t]] = [[[x, y], z], t] + [[[y, z], t], x] + [[[z, t], x], y] + [[[t, x], y], z]$ for all $x, y, z, t \in A$ (see [3]). From this identity, it is easily seen that $K_2(A)$ is an ideal of A^-. Owing to the known classification of finite-dimensional simple alternative algebra over F [5, 6], we find that A is isomorphic either to the split octonion algebra C over F or to the $n \times n$ matrix algebra $M(n, F)$ over F. Let C_0 and $sl(n, F)$ be the sets of trace zero elements in C and $M(n, F)$. Then, it easily follows that $F1$, C_0 and $sl(n, F)$ are the only proper ideals of C^- and $M(n, F)^-$. Thus, it must be that $K_2(A) = F1$ in either case. Since $Z(A) = F1$, we have that $Z(A) = K_2(A)$.

If $p + q \neq 0$ then it follows from Theorem 3 (iii) that $Z(A)e = Z(A(p, q))$, and hence part (i) is proved in this case. Suppose then that $p + q = 0$. Thus, $e = (p - q)^{-1} = (2p)^{-1} = (-2q)^{-1}$, and so p and q are invertible in A. The product $x * y$ in $A(p, q)$ is given by $x * y = xpy + yqx$ and hence $A(p, q) \simeq A(p)^+$, where as in the proof of Theorem 2 $A(p)$ is the algebra with multiplication $x \cdot y = xpy$ defined on the vector space A ($A(p)$ is called the p-isotope of A). Since the map $g : A \to A(p)$ defined by $g(x) = xp^{-1}$ is an algebra isomorphism, $A(p)$ is an alternative algebra also. Letting $(x, y, z)^+$ denote the associator in A^+, we have from [6, p. 53] the identity

$$(x, y, z)^+ = -2(x, y, z) + [y, [x, z]] \tag{7}$$

holding for any alternative algebra A. Note first that $g((x, y, z)^+) = (x, y, z)^*$, the associator in $A(p, q)$, since $A(p, q) \simeq$ A$(p)^+$. If $xp \in Z(A)$, so that $x \in Z(A)e$, then by (7) $g((x, A, A)^+) = g((A, x, A)^+) = g((A, A, x)^+) = 0$ and so $x \in N(A(p, q))$. Since $K(A(p, q)) = A(p, q)$

by Theorem 3 (i), this shows that $Z(A)e \subset Z(A(p, q))$. Conversely, let $x \in N(A(p, q))$. Then, $g^{-1}(x) = xp$ is in $N(A^+)$ and hence by (7)

$$(xp, u, v) = \frac{1}{2}[u, [xp, v]] \tag{8}$$

for all $u, v \in A$. Since A is alternative, when $u = xp$ in (8), we have $[xp, [xp, v]] = 0$ for all $v \in A$. But by our assumption, $A = C$ or $A = M(n, F)$ which implies that xp is in $F1 = Z(A)$. Thus, $x \in Z(A)e$ and $N(A(p, q)) \subset Z(A)e$, so $Z(A)e = Z(A(p, q))$. Therefore, we have established part (i). Part (ii) follows immediately from part (i) and Theorem 2.

Theorem 4 has been proved in [1] for the associative case.

3. Semisimplicity in $A(p, q)$

For a power-associative algebra B, an element x in B is called nilpotent if $x^n = 0$ for some $n > 0$. A nil ideal of B is an ideal of B in which every element is nilpotent. There exists a unique maximal nil ideal of B, which is defined to be the *nilradical* of B. We denote the nilradical by $NR(B)$. If $NR(B) = 0$ then B is said to be (nil) *semisimple.*

We retain the assumptions that A denotes a finite-dimensional alternative algebra over a field F of characteristic $\neq 2, 3$ with multiplication denoted by xy and p, q are in $N(A)$. Thus, as is well known [5, 6], the nilradical $NR(A)$ of A coincides with the solvable radical of A which is shown to be nilpotent also. In fact, $NR(A)$ equals several other radicals (see [5, 6]). Note also that if A is semisimple then A is a direct sum of simple ideals in A[5]. As before, assuming that $A(p, q)$ has a unit element e, by Theorem 1(v) $A(p, q)$ is power-associative, so that $NR(A(p, q))$ is definable. The principal result in this section is to show that $NR(A(p, q)) = NR(A)$, and hence A is semisimple if and only if $A(p, q)$ is semisimple. We proceed as for the associative case [1].

LEMMA 5. *Assume that $A(p, q)$ has a unit element e.*

(i) *If $A = A_1 \oplus \cdots \oplus A_n$ is a direct sum of ideals A_i in A then $A(p, q) \simeq A_1(p_1, q_1) \oplus \cdots \oplus A_n(p_n, q_n)$ and each $A_i(p_i, q_i)$ has a unit element e_i, where p_i, q_i, e_i are the A_i-components of p, q, e.*

(ii) *If I is an ideal of A (so an ideal of $A(p, q)$) then $A(p, q)/I$ is isomorphic to the $(p+I, q+I)$-mutation $(A/I)(p+I, q+I)$ of A/I.*

Proof. (i) Since $A_iA_j=0$ for $i\neq j$, if $x=x_1+\cdots+x_n$ with $x_i\in A_i$ then $x*y=xpy-yqx=\sum_{i=1}^{n}(x_ip_iy_i-y_iq_ix_i)$. This gives the desired isomorphism, and clearly each e_i is a unit element of $A_i(p_i, q_i)$. Part(ii) is straightforward.

Extending a result in [1] for the associative case, we prove

THEOREM 6. *Assume that $A(p, q)$ has a unit element e. Then,*
(i) $NR(A)=NR(A(p, q))$.
(ii) *A is semisimple if and only if $A(p, q)$ is semisimple.*

Proof. Assume first that A is semisimple. Then, A is a direct sum $A=A_1\oplus\cdots\oplus A_n$ of simple ideals A_i in A [5]. By Lemma 6(i), $A(p, q)\simeq A_1(p_1, q_1)\oplus\cdots\oplus A_n(p_n, q_n)$, where each $A_i(p_i, q_i)$ is simple by Theorem 2. Thus, $A(p, q)$ is semisimple also. Next, we show that $NR(A)\subset NR(A(p, q))$. Let x be any element of $NR(A)$. Then, $x(p-q)\in NR(A)$ and hence $(x(p-q))^n=0$ for some $n>0$. Let x^{*m} denote the mth power in $A(p, q)$. Then, $x^{*}=x(p-q)x$ and by induction on m we have $x^{*m}=[x(p-q)]^{m-1}x$ for $m\geq 1$ using (2). Hence, $x^{*(n+1)}=0$, and this shows that $NR(A)$ is a nilideal of $A(p, q)$, which must be contained in $NR(A(p, q))$.

Let $I=NR(A)$. Since A/I is semisimple, $(A/I)(p+I, q+I)$ is semisimple also. But, by Lemma 5 (ii) $(A/I)(p+I, q+I)$ is isomorphic to $A(p, q)/I$ and hence $A(p, q)/I$ is semisimple. Since $I\subset NR(A(p, q))$, $NR(A(p, q))/I$ is a nil ideal of $A(p, q)/I$ and so $NR(A(p, q))/I=0$. Therefore, we have established $I=NR(A(p, q))$, showing part(i). Part (ii) is an immediate consequence of part (i).

REMARKS. It is not known whether $NR(A(p, q))$ coincides with the solvable radical of $A(p, q)$. However, we conjecture that this is the case. In relation to Theorem 2, it is shown in [1] that if A is an associative algebra, then A is prime if and only if $A(p, q)$ is prime. For an alternative algebra, this is an open problem.

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