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GENERALIZATIONS OF H-GROUPS AND CO H-GROUPS

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1. The Ex-homotopy Theory

References for the ex-homotopy theory are [3] and [4]. We review some of these results in this section.

Let B be a space. By an *ex-space(over B)* we mean a triple (X, σ, ρ), where X is a space and

$$B \xrightarrow{\sigma} X \xrightarrow{\rho} B$$

are maps such that $\rho\sigma=1$. Normally it will be sufficient to denote the ex-space by X. We refer to σ as the section, to ρ as the projection. Together they constitute an ex-structure on the total space X over the base space B. Notice that B can always be regarded as an ex-space over itself, with $\sigma=1=\rho$. We refer to this as the *trivial ex-space* over the given base.

We describe an ex-space as *proper* if σB is a closed subspace of X. When this condition is satisfied we can embed B in X, by means of σ, so that ρ constitutes a retraction. Instead of regarding B as a retract of X, we regard X as an "extract" of B. This change of view opens up the prospect of the following development.

We shall outline a theory which reduces to ordinary homotopy theory when B is a point. The generalization proceeds on formal lines, for the most part ; whenver we meet a base point, in the ordinary theory, we replace it by B, using the section and projection in an appropriate way.

Starting from the given base space, I. M. James had begun to construct a new category out of the category of topological spaces. The objects in the new category are ex-spaces. The morphism is defined as follows ;

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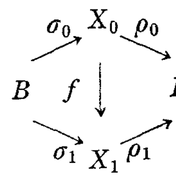
Let $X_i(i=0,1)$ be an ex-space with section σ_i and projection ρ_i. By an *ex-map* $f: X_0 \to X_1$, we mean an ordinary map such that

$$f\sigma_0=\sigma_1, \quad \rho_1 f=\rho_0,$$

as shown in the following diagram :



By the *trivial ex-map* of X_0 to X_1 we mean $\sigma_1\rho_0$, which satisfies the ex-map condition since $\rho_i\sigma_i=1$.

Note that the composition of any ex-map with a trivial ex-map, on either side, is again a trivial ex-map. Thus the category of ex-spaces and ex-maps is a pointed category. The equivalence in this category will be called *ex-homeomorphism.*

In this category, products are defined as follows ;

Let $X_i(i=0,1)$ be an ex-space over B. The *direct product* $X_0 \times X_1$ is the subspace of the ordinary topological product consisting of pairs (x_0, x_1) such that $\rho_0 x_0=\rho_1 x_1$, with the section σ and the projection ρ given by

$$\sigma(b)=(\sigma_0(b),\ \sigma_1(b)), \qquad (b\in B)$$
$$\rho(x_0,x_1)=\rho_0(x_0)=\rho_1(x_1), \qquad (x_i\in X_i).$$

The *inverse product*, or *wedge sum*, $X_0 \vee X_1$ can be defined as the subspace of the direct product consisting of pairs (x_0, x_1) such that

$$x_0=\sigma_0\rho_1(x_1) \text{ or } x_1=\sigma_1\rho_0(x_0).$$

Structural ex-maps

$$X_0 \longleftarrow X_0 \times X_1 \longrightarrow X_1,$$
$$X_0 \longrightarrow X_0 \vee X_1 \longleftarrow X_1$$

are defined in the obvious way. *Direct* and *inverse product* of ex-maps are similarly defined.

Let X, Y be ex-spaces, over B. By an *ex-homotopy*

$$f_t : X \longrightarrow Y \qquad (t\in I)$$

we mean an ordinary homotopy such that f_t is an ex-map for all values of t. We write $f_0 \simeq f_1$, when such an ex-homotopy exists. This defines an equivalence relation, on the set of ex-maps of X into Y, and we denote

$\pi(X, Y)$, the set of ex-homotopy classes thus obtained. The class of the trivial ex-map is denoted by 0. As in the ordinary homotopy theory, $[f]$ will be denoted the ex-homotopy class which contains f.

Let Z be an ex-space, over B. Composition on the left with an ex-map $g : Y \longrightarrow Z$ determines a function

$$g_* : \pi(X, Y) \longrightarrow \pi(X, Z),$$

while composition on the right with an ex-map $f : X \longrightarrow Y$ determines a function

$$f^* : \pi(Y, Z) \longrightarrow \pi(X, Z).$$

Both functions send 0 to 0.

Let $\pi_i (i=0, 1)$ denote the structural ex-maps of the direct product of ex-spaces X_i : thus

$$X_0 \xleftarrow{\pi_0} X_0 \times X_1 \xrightarrow{\pi_1} X_1.$$

An ex-map h : $X \longrightarrow X_0 \times X_1$ determines, and is determined by its component ex-maps $(\pi_0 h, \ \pi_1 h)$, where $\pi_i h : X \longrightarrow X_i$, and ex-homotopies behave in the same way. Thus we obtained a natural equivalence

$$\pi(X, X_0 \times X_1) \longleftrightarrow \pi(X, X_0) \times \pi(X, X_1).$$

Similarly in the case of the inverse product we obtaian a natural equivalence

$$\pi(X_0 \vee X_1, X) \longleftrightarrow \pi(X_0, X) \times \pi(X_1, X).$$

We say that an ex-map $f : X \longrightarrow Y$ is an *ex-homotopy equivalence* if there is an ex-map $g : Y \longrightarrow X$ such that

$$fg \simeq 1, \quad gf \simeq 1.$$

When such a pair of ex-maps exists we say that X and Y have the *same ex-homotopy type*. If X has the same ex-homotopy type as the base space B, we say that X is *ex-contractible*.

2. Ex H-groups and Ex H-cogroups

Here and in the sequel, we assume that all spaces are locally compact, T_2 and proper ex-spaces.

DEFINITION 2.1. Let (X, σ, ρ) be an ex-space over B. The *diagonal map* $\Delta_X : X \longrightarrow X \times X$ is defined by $\Delta_X(x) = (x, \ x)$, and the *folding* *map* $\nabla_X : X \vee X \longrightarrow X$ is also defined by $\nabla_X(x, \sigma\rho x) = x$ and $\nabla_X(\sigma\rho x, x) = x$. We can easily show that these maps are ex-maps.

PROPOSITION 2.1. *Let* X, X' *and* Y *be ex-spaces over* B. *Then the inverse and direct products of ex-maps have the following properties;*

(*a*) *If* $f \simeq f', g \simeq g' : X \longrightarrow Y$, *then* $f \times g \simeq f' \times g'$ *and* $f \vee g \simeq f' \vee g'$.

(*b*) *If* $f : X \longrightarrow Y$, *then* $f\nabla_X = \nabla_Y(f \vee f)$ *and* $(f \times f)\Delta_X = \Delta_Y f$.

(*c*) *If* $f, g : X \longrightarrow Y$ *and* $f', g' : X' \longrightarrow X$, *then*
$(f \times g) \circ (f' \times g') = f \circ f' \times g \circ g'$ *and*
$(f \vee g) \circ (f' \vee g') = f \circ f' \vee g \circ g'$.

DEFINITION 2.2. An ex-space Y is called an *ex H-space* if there exists an ex-map $m : Y \times Y \longrightarrow Y$ such that

$$m(1_Y \times \sigma_Y \rho_Y)\Delta_Y \simeq 1_Y \simeq m(\sigma_Y \rho_Y \times 1_Y)\Delta_Y.$$

Note that if B is a point, then Y is an H-space iff Y is an ex H-space.

An ex H-space Y is said to be *associative* if

$$m(m \times 1_Y) \simeq m(1_Y \times m) : Y \times Y \times Y \longrightarrow Y,$$

and an *inverse* is an ex-map $u : Y \longrightarrow Y$ such that

$$m(u \times 1_Y)\Delta_Y \simeq m(1_Y \times u)\Delta_Y \simeq \sigma_Y \rho_Y.$$

We will say that Y is an *ex H-group* if it is an associative ex H-space with an inverse.

THEOREM 2.2. *If* X *is any ex-space and* Y *is an ex H-group, then* $\pi(X, Y)$ *can be given the structure of a group.*

Proof. Given two ex-maps $f, g : X \longrightarrow Y$, let $f \cdot g$ be $m(f \times g)\Delta_X$, this is certainly another ex-map from X to Y. Moreover, given further maps f', $g' : X \longrightarrow Y$, such that $f \simeq f'$, $g \simeq g'$, then $f \cdot g \simeq f' \cdot g'$ by Proposition 2.1, and so a multiplication in $\pi(X, Y)$ can be unambiguously defined by $[f] \cdot [g] = [f \cdot g]$.

It remains to show that this multiplication satisfies the axioms for a group. First, given a third ex-map $h : X \longrightarrow Y$, we have

$$\begin{aligned}(f \cdot g) \cdot h &= m(f \cdot g \times h)\Delta_X \\ &= m(m(f \times g)\Delta_X \times h)\Delta_X \\ &= m(m \times 1)((f \times g) \times h)(\Delta_X \times 1_X)\Delta_X \\ &\simeq m(1 \times m)(f \times (g \times h))(1_X \times \Delta_X)\Delta_X \\ &= f \cdot (g \cdot h).\end{aligned}$$

So that $([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h])$.

Secondly

$$\begin{aligned} f\cdot\sigma_Y\rho_X &= m(f\times\sigma_Y\rho_X)\Delta_X \\ &= m(1\times\sigma_Y\rho_Y)(f\times f)\Delta_X \\ &= m(1\times\sigma_Y\rho_Y)\Delta_Y f \\ &\simeq 1_Y f, \\ &= f \end{aligned}$$

and similarly we obtain $\sigma_Y\rho_X\cdot f\simeq f$, so that $0=[\sigma_Y\rho_X]$ is a unit element for $\pi(X, Y)$.

Finally, we may define $[f]^{-1}=[uf]$, since

$$\begin{aligned} uf\cdot f &= m(uf\times f)\Delta_X \\ &= m(u\times 1)(f\times f)\Delta_X \\ &= m(u\times 1)\Delta_Y f \\ &\simeq \sigma_Y\rho_Y f \\ &= \sigma_Y\rho_X \end{aligned}$$

and similarly $f\cdot uf\simeq\sigma_Y\rho_X$.

Naturally also, maps of X give rise to homomorphisms, not just functions.

Theorem 2.3. *If $g : X_0 \longrightarrow X_1$ is an ex-map, and Y is an ex H-group, then $g^* : \pi(X_1, Y) \longrightarrow \pi(X_0, Y)$ is a homomorphism. In particular, g^* is an isomorphism if g is an ex-homotopy equivalence.*

Proof. Given ex-maps $f, f' : X_1 \longrightarrow Y$, we have

$$\begin{aligned} (f\cdot f')g &= m(f\times f')\Delta_{X_1}g \\ &= m(f\times f')(g\times g)\Delta_{X_0} \\ &= (fg\times f'g)\Delta_{X_0} \\ &= (fg)\cdot(f'g). \end{aligned}$$

Thus $g^*([f]\cdot[f'])=g^*([f])\cdot g^*([f'])$.

Example. The trivial ex-space B is an ex H-group. For, if we define $m : B\times B \longrightarrow B, u : B \longrightarrow B$ by

$$m(b, b)=b, \quad u=1_B,$$

then these maps m and u are ex-maps and clearly are the ex H-group structure. We will call this B the *trivial ex H-group*.

Before giving examples of non-trivial ex H-groups, let us examine the 'dual' situation, in which $\pi(X, Y)$ becomes a group because of the properties possessed by X rather than Y.

First, we can have

THEOREM 2.4. $X\vee(Y\vee Z)\equiv(X\vee Y)\vee Z$.

Proof. Since $h:\ (X\times Y)\times Z\equiv X\times(Y\times Z)$, $h((a,b),c)\equiv(a,(b,c))$, it suffices to show the restriction of h on $(X\vee Y)\vee Z$ is well defined. If $((a,b),\ c)\in(X\vee Y)\vee Z$, then we must treat four cases.

Case I. Suppose that

$$b=\sigma_Y\rho_X(a),\ \ c=\sigma_Z\rho_Y(b)=\sigma_Z\rho_X(a).$$

Then

$$(b,c)=(\sigma_Y\rho_X(a),\ \ \sigma_Z\rho_X(a)),\ \ c=\sigma_Z\rho_Y(b),$$

so that $(a,(b,c))\in X\vee(Y\vee Z)$.

Case II. Suppose that

$$a=\sigma_X\rho_Y(b),\ \ c=\sigma_Z\rho_X(a)=\sigma_Z\rho_Y(b).$$

Then the desired conclusion holds also in this case.
We omit the proof for the remaining cases.

COROLLARY 2.5. $\nabla_X(\nabla_X\vee 1)=\nabla_X(1\vee\nabla_X)h$, *where* $h:\ (X\vee X)\vee X\longrightarrow X\vee(X\vee X)$ *is the ex-homeomorphism defined in the previous Theorem* 2.4.

DEFINITION 2.3. An ex-space X is called an *ex co H-space* if there exists an ex-map $m:X\longrightarrow X\vee X$ such that

$$\nabla_X(1_X\vee\sigma\rho)m\simeq 1_X\simeq\nabla_X(\sigma\rho\vee 1_X)m.$$

Note that if B is a point, then X is a co H-space iff it is an ex co H-space.

An ex co H-space X is said to be *associative* if

$$(m\vee 1)m\simeq(1\vee m)m:X\longrightarrow X\vee X\vee X,$$

and an *inverse* is an ex-map $v:X\longrightarrow X$ such that

$$\nabla_X(v\vee 1)m\simeq\nabla_X(1\vee v)m\simeq\sigma_X\rho_X.$$

Again, we shall say that X is an *ex co H-group* (or *ex H-cogroup*) if it is an associative ex co H-space with an inverse.
Notice that the definition of an ex co H-space(group) closely resembles that of an ex H-space(group) : we merely turn all the maps round and use the inverse product instead of the direct product.

THEOREM 2.6. *If X is an ex co H-group and Y is any ex-space, $\pi(X,Y)$ can be given the structure of a group. Moreover, if $g:Y\longrightarrow Z$ is an ex-map, $g_*:\pi(X,Y)\longrightarrow\pi(X,Z)$ is a homomorphism, and* *so is an isomorphism if g is an ex-homotopy equivalence.*

Proof. Given two ex-maps $f, g: X \longrightarrow Y$, define

$$f+g=\nabla_Y(f\vee g)m.$$

Then '+' is well defined in $\pi(X, Y)$ by Proposition 2.1.
As in the proofs of Theorem 2.2. and 2.3, we can proceed the proof by using Corollary 2.5.

To give an example of an ex co H-group, we introduce the notion of suspension defined by I. M. James [3]. Let X be a proper ex-space, over B. By the *suspension* of X, in the ex-category, we mean the ex-space (SX, σ', ρ') defined as follows. Consider the ordinary cylinder $X \times I$, and write

$$\pi(x, t)=\rho x \quad (x\in X, \ t\in I).$$

Then SX is obtained from $X\times I$ by identifying points of $B\times I \cup X\times \dot{I}$ which have the same image under π. The section σ' is given by $\sigma' b=(b, t)$, for any t, and the projection ρ' is induced by π. Note that SX is a proper ex-space.

Suspension of ex-maps and ex-homotopies is similarly defined. Now form the wedge sum of SX with itself and consider the structural ex-maps

$$SX \xrightarrow[u_0]{} SX\vee SX \xleftarrow[u_1]{} SX$$

There is an ex-map $m: SX \longrightarrow SX\vee SX$, defined by

$$m(x, t)=\begin{cases} u_0(x, 2t), & \left(0\leqq t\leqq \frac{1}{2}\right) \\ u_1(x, 2t-1), & \left(\frac{1}{2}\leqq t\leqq 1\right). \end{cases}$$

Using the natural identification,

$$\pi(SX\vee SX, Y)\cong\pi(SX, Y)\times\pi(SX, Y),$$

where Y is any ex-space, we can regard the induced function m^* as a binary operation on the set $\pi(SX, Y)$. We call this operation *track addition* and normally write

$$m^*([f], \ [g])=[f]+[g] \quad ([f], [g]\in\pi(SX, Y))$$

without meaning to suggest that the operation is commutative. Note that $m^*([f], \ [g])=[\nabla_Y(f\vee g)m]$. The formal properties of m are the same as in the ordinary theory. Thus the ex-homotopy associativity property is established, by the same argument as in the ordinary case, and it follows that the track addition is associative. Similarly for the other properties, so that we can finally obtain that SX is an ex co H-group.

COROLLARY 2.7. [3] *Under the track addition the set* $\pi(SX, Y)$ *forms a group. If* X *has the same ex-homotopy type as* SX', *for some ex-space* X', *then the group is abelian.*

3. Suspensions and Loop spaces

Let X be a proper, locally compact T_2 ex-space, over B. We embed B in X by means of the section σ_X. Let X^I be the collection of mappings of I into X with the compact-open topology.

The *loop space* ΩX of X is the subspace of X^I characterized by $w\in \Omega X$ iff $w(0)=w(1)\in B\subset X$. Define section σ and projection ρ as follows :

$$\sigma(b)=\tilde{b},\ \ \tilde{b}(t)=b \qquad (t\in I)$$
$$\rho(w)=w(0)\in B, \qquad (w\in\Omega X)$$

The continuity of σ and ρ follows from a standard argument in the compact-open topology. Moreover $\rho\sigma=1_B$, so that ΩX is an ex-space. Note that ΩX is a proper ex-space.

Define a map $m : \Omega X\times\Omega X \longrightarrow \Omega X$ by

$$m(w, w')(t)=\begin{cases} w(2t), & \left(0\leqq t\leqq \frac{1}{2}\right) \\ w'(2t-1), & \left(\frac{1}{2}\leqq t\leqq 1\right). \end{cases}$$

Since $(w, w')\in\Omega X\times\Omega X$, $w(1)=w(0)=\rho(w)=\rho(w')=w'(0)=w'(1)$, so that m is well defined. The continuity of m also follows from a standard argument of the compact-open topology and pasting lemma. Clearly m is an ex-map.

THEOREM 3.1. *ΩX is an ex H-group.*

Proof. (First step). $m(1_{\Omega X}\times\sigma\rho)\Delta_{\Omega X}\simeq 1_{\Omega X}\simeq m(\sigma\rho\times 1_{\Omega X})\Delta_{\Omega X}$. Define a map $H : \Omega X\times I\times I \longrightarrow X$ by

$$H(w, s, t)=\begin{cases} w\left(\frac{2s}{2-t}\right), & \left(0\leqq s\leqq \frac{2-t}{2}\right) \\ \sigma\rho w(2s-(2-t)), & \left(\frac{2-t}{2}\leqq s\leqq 1\right) \end{cases}$$

Then H is well defined and continuous, so that the adjoint map $\tilde{H}$:

$\Omega X\times I \longrightarrow \Omega X$ defined by $\tilde{H}(w, t)(s)=H(w, s, t)$ is well defined and continuous. This $\tilde{H}$ is an ex-homotopy between $1_{\Omega X}$ and $m(1_{\Omega X}\times\sigma\rho)\Delta_{\Omega X}$. Similarly we can obtain

$$1_{\Omega X}\simeq m(\sigma\rho\times 1_{\Omega X})\Delta_{\Omega X}.$$

(Second step). $m(m\times 1_{\Omega X})\simeq m(1_{\Omega X}\times m)$.

As in the ordinary theory, define a map

$$H:\Omega X\times\Omega X\times\Omega X\times I\times I \longrightarrow X$$

by

$$H(w, w', w'', s, t)=\begin{cases} w\left(\dfrac{4s}{1+t}\right), & \left(0\leqq s\leqq\dfrac{1+t}{4}\right)\\ w'(4s-t-1), & \left(\dfrac{1+t}{4}\leqq s\leqq\dfrac{2+t}{4}\right)\\ w''\left(\dfrac{4s-t-2}{2-t}\right), & \left(\dfrac{2+t}{4}\leqq s\leqq 1\right).\end{cases}$$

This map H is well defined and continuous. So that the adjoint map $\tilde{H}:\Omega X\times\Omega X\times\Omega X\times I \longrightarrow \Omega X\subset X^I$ defined by $\tilde{H}(w, w', w'', t)(s)=H(w, w', w'', s, t)$ is the required ex-homotopy.

(Final Step). $m(u\times 1_{\Omega X})\Delta_{\Omega X}\simeq\sigma\rho\simeq m(1_{\Omega X}\times u)\Delta_{\Omega X}$.

Now we define an inverse ex-map $u:\Omega X \longrightarrow \Omega X$ by

$$u(w)(t)=w(1-t).$$

Let $K:\Omega X\times I\times I \longrightarrow X$ be the map defined by

$$K(w, s, t)=\begin{cases} w(1-2st), & \left(0\leqq s\leqq\dfrac{1}{2}\right)\\ w((2s-1)t+1-t), & \left(\dfrac{1}{2}\leqq s\leqq 1\right).\end{cases}$$

Clearly K is well defined and continuous and the adjoint map $\tilde{K}:\Omega X\times I \longrightarrow \Omega X$ defined by $\tilde{K}(w, t)(s)=K(w, s, t)$ becomes an ex-homotopy between $\sigma\rho$ and $m(u\times 1_{\Omega X})\Delta_{\Omega X}$. Similarly we can have $\sigma\rho\simeq m(1_{\Omega X}\times u)\Delta_{\Omega X}$. This completes the proof.

Note that ΩX is a non-trivial ex H-group.

THEOREM 3.2. *If Y is an ex H-group, then two multiplications in $\pi(SX, Y)$ are the same, and they are commutative.*

Proof. Let $f, g: SX \longrightarrow Y$ be ex-maps. Then their track addition "induced by the suspension structure" is given by

$$(f+g)(x,t)=\begin{cases} f(x,2t), & \left(0\leqq t\leqq \frac{1}{2}\right) \\ g(x,2t-1), & \left(\frac{1}{2}\leqq t\leqq 1\right). \end{cases}$$

Their product "induced by the ex H-group structure" is given by

$$(f\cdot g)(x,t)=m(f(x,t),g(x,t)).$$

These two multiplications are "independently defined", and "commute" with each other. This is expressed by the formula

$$(f\cdot g)+(f'\cdot g')=(f+f')\cdot(g+g').$$

In fact both sides of this equation are given by the formula

$$h(x,t)=\begin{cases} m(f(x,2t),\ g(x,2t)), & \left(0\leqq t\leqq \frac{1}{2}\right) \\ m(f'(x,2t-1),\ g'(x,2t-1)), & \left(\frac{1}{2}\leqq t\leqq 1\right). \end{cases}$$

Letting $g=f'=\sigma_Y\rho_{SX}$, this formula reduces to

$$f+g'=f\cdot g',$$

hence the multiplications are equal. Letting $f=g'=\sigma_Y\rho_{SX}$, this reduces to

$$g+f'=f'\cdot g,$$

hence the multiplication is commutative.

THEOREM 3.3. *Let X, Y be proper ex-spaces. Then there is a monomorphism $\phi:\pi(SX,Y)\longrightarrow\pi(X,\Omega Y)$.*

Proof. Let $f:SX\longrightarrow Y$ be an ex-map. Let $q:X\times I\longrightarrow SX$ be the quotient map. Define $\phi:\pi(SX,Y)\longrightarrow\pi(X,\Omega Y)$ by

$$\phi([f])=[\widetilde{fq}],$$

where $\widetilde{fq}:X\longrightarrow \Omega Y\subset Y^I$ be the adjoint map of fq.
Note that if $f\simeq f'$, then $\widetilde{fq}\simeq\widetilde{f'q}$. Since f is an ex-map, we can easily prove that $\widetilde{fq}$ is an ex-map. Thus ϕ is well defined.

Now we will prove ϕ is a homomorphism.
Let $f,g:SX\longrightarrow Y$. We will prove that

$$\widetilde{(f+g)q}=\widetilde{fq}\cdot\widetilde{gq}.$$

For all $x\in X$ and $t\in I$, we have

$$(((f+g)q)(x))(t)=\begin{cases} f(x,2t), & \left(0\leqq t\leqq \frac{1}{2}\right) \\ g(x,2t-1), & \left(\frac{1}{2}\leqq t\leqq 1\right). \end{cases}$$

$$((\widetilde{fq}\cdot\widetilde{gq})(x))(t)=\begin{cases} f(x,2t), & \left(0\leqq t\leqq \frac{1}{2}\right)\\ g(x,2t-1), & \left(\frac{1}{2}\leqq t\leqq 1\right)\end{cases}$$

This equality implies that ϕ is a homomorphism.

Finally let $\widetilde{fq}\simeq\widetilde{gq}$, i.e. let $F: X\times I\longrightarrow \Omega Y$ be an ex-homotopy between $\widetilde{fq}$ and $\widetilde{gq}$. Define an ex-homotopy $H: SX\times I\longrightarrow Y$ by

$$H(x,t,s)=F(x,s)(t).$$

Then since F is an ex-homotopy, we can easily show that H is a well defined ex-homotopy between f and g. This completes the proof.

Remark. In general, the homomorphism ϕ defined in the previous Theorem is not surjective. Indeed, let the unit circle S^1 be the trivial ex-space B. Since π $(S\Omega B,\ B)=\{0\}$, we will prove that π $(\Omega B,\Omega B)\neq\{0\}$. Define $u:\Omega B\longrightarrow\Omega B$ by $u(w)(t)=w(1-t)$, $t\in I$. Then $[u]$ is the non-trivial element of $\pi(\Omega B,\ \Omega B)$. For, if $u\simeq\sigma\rho$, then there exists an ex-homotopy $H:\Omega B\times I\longrightarrow\Omega B$ such that

$$H(w,0)(s)=\widetilde{w(0)}(s),$$
$$H(w,1)(s)=w(1-s).$$

Define a map $K: I\times I\longrightarrow B$ by $K(s,t)=H(w,t)(s)$, where w is a fixed path in B which is not homotopic to the constant path. Then K is a homotopy between w^{-1} and the constant path $\widetilde{w(0)}$. So that $u\not\simeq\sigma\rho$.

Conjecture. What kinds of ex-spaces X, Y can have ϕ as an isomorphism?

Denote $S^nX=S(S^{n-1}X)$ $n\geqq 2$. Then we have

Corollary 3.4. *$\pi(S^2X, Y)$ is abelian* (cf. Corollary 2.7).

Proof. Since $\phi:\pi(S^2X,Y)\longrightarrow\pi(SX,\Omega Y)$ is a monomorphism and $\pi(SX,\ \Omega Y)$ is abelian, we can easily find that

$$f+g=g+f.$$

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