J. Korean Math. Soc. 24(1987), No. 1, pp. 83-89

NONEMBEDDABILITY AND NONIMMERSIBILITY OF A PRODUCT LENS SPACE

MOO YOUNG SOHN

1. Introduction

In [1], M. F. Atiyah introduced a method to solve non-immersion and nonembedding problems of a smooth manifold M by using Adams operators on the Grothendieck ring $\widetilde{KO}(M)$, and he applied his method to the case of the real projective space $RP(n)$. Later, H. Suzuki [8] estimated a lower bound of i, j such that $RP(n)\times RP(m)$ can be immersed in R^{n+m+i} and embedded in R^{n+m+j}, and M. Yasuo [9] considered the case of product lens space $\prod_{i\in I} L^{2n_i+1}(p^{m_i})$. In this paper, making use of the method initiated by [1], [8], [9], we estimate a upper bound of the number of linearly independent tangent vector fields over a product lens space $L^{2n+1}(p)\times L^{2m+1}(q)$, where p, q are any odd prime numbers, and a lower bound of k, l such that $L^{2n+1}(p)\times L^{2m+1}(q)$ can be immersed in $R^{2(n+m+1)+k}$ and embedded in $R^{2(n+m+1)+l}$. In what follows, M will mean a smooth closed manifold. Immersion and embedding will mean C^∞-differentiable ones.

2. γ-operator over the Grothendieck ring

Let F denote either the real field $\boldsymbol{R}$ or the complex field $\boldsymbol{C}$, and let $\mathrm{Vect}_F(M)$ denote the set of equivalence classes of F-vector bundles over M. The Whitney sum of vector bundles makes $\mathrm{Vect}_F(M)$ a semi-group and the Grothendieck group $K_F(M)$ is the associated abelian group. The tensor product of vector bundles defines a commutative ring structure in $K_F(M)$. As usual, we use the notation $KO(M)$ and $K(M)$ for $K_R(M)$ and $K_C(M)$ respectively. The trivial bundle of dimension n will simply be denoted by n. Let x_0 be a base point of M, then clearly $KO(x_0)=\boldsymbol{Z}$ (the ring of integers).

Received October 30, 1986.

We define $\widetilde{KO}(M)=\mathrm{Ker}\{i^*: KO(M)\to \boldsymbol{Z}\}$, where i^* is the homomorphism induced by the natural inclusion $\{x_0\}\to M$, then clearly $KO(M)\approx \boldsymbol{Z}\oplus\widetilde{KO}(M)$.

For $x\in \mathrm{Vect}_R(M)$, the vector bundle $\lambda^i(x)$ is defined by the exterior power operation $\wedge^i(x)$, $i=0, 1, 2, 3, \cdots$.
We get the following formal properties of the operation λ^i.

(1) $\lambda^0(x)=1$ (2) $\lambda^1(x)=x$

(3) $\lambda^i(x+y)=\sum_{i=0}^{j}\lambda^j(x)\lambda^{i-j}(x)$ (4) $\lambda^i(x)=0$ for $i>\dim x$.

We define $\lambda_t(x)=\sum_{i=0}^{\infty}\lambda^i(x)t^i$, where t is an indeterminate. Let $A(M)$ denote the multiplicative group of formal power series in t with coefficient in $KO(M)$ and with constant term 1. Then (1) and (3) assert that λ_t defines a homomorphism $\mathrm{Vect}_R(M)\to A(M)$. Hence we get a homomorphism $\lambda_t: KO(M)\to A(M)$ and operators $\lambda^i: KO(M)\to KO(M)$ with $\lambda_t(x)=\sum_{i=0}^{\infty}\lambda^i(x)t^i$. The γ-operation in $KO(M)$, $\gamma_t: KO(M)\to A(M)$, is defined by the requirement that $\gamma_t(x)=\lambda_{t/1-t}(x)$ and $\gamma_t(x)=\sum_{i\geqq 0}\gamma^i(x)t^i$ for $x\in KO(M)$.

Now let $\tau(M)$ denote the tangent bundle over M and put $\tilde{\tau}(M)=\tau(M)-n\in\widetilde{KO}(M)$, then the operations γ^i give us an information of the structure of tangent bundle over M as follows.

THEOREM 2.1([8]). *If $\gamma^i(\tilde{\tau}(M))\neq 0$ for an i, $0\leqq i\leqq n$, then the number of linearly independent tangent vector fields over M does not exceed $n-i$.*

The following Atiyah criterion for an immersion and an embedding will be used for our main result.

THEOREM 2.2([1]). *If M is immersed in the $(n+k)$-dimensional euclidean space R^{n+k}, then we have $\gamma^i(-\tilde{\tau}(M))=0$ for $i>k$. If M is embedded in R^{n+l}, then we have $\gamma^i(-\tilde{\tau}(M))=0$ for $i\geqq l$.*

The γ-dimension and γ-codimension of an n-dimensional manifold M are defined as follows;

$$\mathrm{Dim}_\gamma(M)=\sup\{i\in\boldsymbol{N}\,|\,\gamma^i(\tau(M)-n)\neq 0\},$$
$$\mathrm{Cod}_\gamma(M)=\sup\{i\in\boldsymbol{N}\,|\,\gamma^i(n-\tau(M))\neq 0\}.$$

Let p, q be any odd prime numbers. $L^{2n+1}(p)=S^{2n+1}/Z_p$, $L^{2m+1}(q)=S^{2m+1}/Z_q$ the standard lens spaces, and let

$$\prod_1 : L^{2n+1}(p)\times L^{2m+1}(q) \to L^{2n+1}(p),$$
$$\prod_2 : L^{2n+1}(p)\times L^{2m+1}(q) \to L^{2m+1}(q),$$
$$\prod_\wedge : L^{2n+1}(p)\times L^{2m+1}(q) \to L^{2n+1}(p)\wedge L^{2m+1}(q)$$

be canonical projections,

The following is easily obtained by using cohomology properties of the Grothendieck ring. where $L^{2n+1}(\ )\wedge\ L^{2m+1}(q)$ is the smash product.

THEOREM 2.3. *(i) The induced homomorphisms*

$$\prod_1{}^* : \widetilde{KO}(L^{2n+1}(p)) \to \widetilde{KO}(L^{2n+1}(p)\times L^{2m+1}(q)),$$
$$\prod_2{}^* : \widetilde{KO}(L^{2n+1}(q)) \to \widetilde{KO}(L^{2n+1}(p)\times L^{2m+1}(q)),$$
$$\prod_\wedge{}^* : \widetilde{KO}(L^{2n+1}(p)\wedge L^{2m+1}(q)) \to \widetilde{KO}(L^{2n+1}(p)\times L^{2m+1}(q))$$

are injective and we have a direct sum decomposition

$$\widetilde{KO}(L^{2n+1}(p)\times L^{2m+1}(q)) = \prod_1{}^*(\widetilde{KO}(L^{2n+1}(p)))$$
$$\oplus \prod_2{}^*(\widetilde{KO}(L^{2m+1}(q))) \oplus \prod_\wedge{}^*(\widetilde{KO}(L^{2n+1}(p)\wedge L^{2m+1}(q))).$$

(ii) If $u\in\widetilde{KO}(L^{2n+1}(p))$ *and* $v\in\widetilde{KO}(L^{2m+1}(q))$, *then*

$$\prod_1{}^*(u)\prod_2{}^*(v)\in\prod_\wedge{}^*(\widetilde{KO}(L^{2n+1}(p)\wedge L^{2m+1}(q)))).$$

3. Applications to a product product lens space

Throughout this section, let p and q be any odd prime numbers, τ the tangent bundle over $L^{2n+1}(p)\times L^{2m+1}(q)$, and $\tilde{\tau}=\tau-2(n+m+1)$. Let ξ,η be the canonical complex line bundles over $L^{2n+1}(p)$ and $L^{2m+1}(q)$ respectively and let $a=\xi-1_C$, $b=\eta-1_C$, where 1_C denote the complex trivial bundle. Then we have the following relations (cf. [3])

$$\tau(L^{2n+1}(p))-(2n+1)=(n+1)\mathrm{re}(a),$$
$$\tau(L^{2m+1}(q))-(2m+1)=(m+1)\mathrm{re}(b),$$
$$\gamma_t(\mathrm{re}(a))=1+\mathrm{re}(a)t-\mathrm{re}(a)t^2,$$
$$\gamma_t(\mathrm{re}(b))=1+\mathrm{re}(b)t-\mathrm{re}(b)t^2,$$

where re denote the realification of a vector bundle.

THEOREM 3.1. $\mathrm{Dim}_\gamma(L^{2n+1}(p)\times L^{2m+1}(q))$

$$=2\ \sup\{k\mid \binom{n+1}{k}(\prod_1{}^*\mathrm{re}(a))^k+\binom{m+1}{k}(\prod_2{}^*\mathrm{re}(b))^k$$
$$+\sum_{\substack{i+j=k\\ 1\leq i\leq n+1\\ 1\leq j\leq m+1}}\binom{n+1}{i}\binom{m+1}{j}(\prod_1{}^*\mathrm{re}(a))^i(\prod_2{}^*\mathrm{re}(b))^j\neq 0\},$$

where $\binom{n}{i}$ *is the binomial coefficient.*

Proof. Let τ_1, τ_2 be the tangent bundles over $L^{2n+1}(p)$, $L^{2m+1}(q)$ respectively, then

$$\begin{aligned}\tilde{\tau} &= \prod{}_1^*\tau_1 + \prod{}_2^*\tau_2 - 2(n+m+1)\\ &= \prod{}_1^*(\tau_1-(2n+1)) + \prod{}_2^*(\tau_2-(2m+1))\\ &= (n+1)\prod{}_1^*(\mathrm{re}(a)) + (m+1)\prod{}_2^*(\mathrm{re}(b)).\end{aligned}$$

By using the property $\gamma_t(x+y)=\gamma_t(x)\gamma_t(y)$ and the naturality of the operator γ_t, we have

$$\begin{aligned}\gamma_t(\tilde{\tau}) &= [\prod{}_1^*\{\gamma_t(\mathrm{re}(a))\}]^{n+1}[\prod{}_2^*\{\gamma_t(\mathrm{re}(b))\}]^{m+1}\\ &= \{1+\prod{}_1^*\mathrm{re}(a)t-\prod{}_1^*\mathrm{re}(a)t^2\}^{n+1}\{1+\prod{}_2^*\mathrm{re}(b)t-\prod{}_2^*\mathrm{re}(b)t^2\}^{m+1}\\ &= \sum_{\substack{0\leq i\leq n+1\\ 0\leq j\leq m+1}}\binom{n+1}{i}\binom{m+1}{j}(\prod{}_1^*\mathrm{re}(a))^i(\prod{}_2^*\mathrm{re}(b))^j(t-t^2)^{i+j}.\end{aligned}$$

If we set

$$\begin{aligned}A_k = &\binom{n+1}{k}(\prod{}_1^*\mathrm{re}(a))^k + \binom{m+1}{k}(\prod{}_2^*\mathrm{re}(b))^k\\ &+\sum_{\substack{i+j=k\\ 1\leq i\leq n+1\\ 1\leq j\leq m+1}}\binom{n+1}{i}\binom{m+1}{j}(\prod{}_1^*\mathrm{re}(a))^i(\prod{}_2^*\mathrm{re}(b))^j,\end{aligned}$$

then, by taking the coefficient of t^i, we have

$$\begin{aligned}&\gamma^0(\tilde{\tau})=1, &&\gamma^1(\tilde{\tau})=A_1,\\ &\gamma^2(\tilde{\tau})=A_2-A_1, &&\gamma^3(\tilde{\tau})=A_3-2A_2,\\ &\gamma^4(\tilde{\tau})=A_4-3A_3+A_2, &&\gamma^5(\tilde{\tau})=A_5-4A_4+3A_3,\\ &\gamma^6(\tilde{\tau})=A_6-5A_5+6A_4-A_3, &&\gamma^7(\tilde{\tau})=A_7-6A_6+10A_5-4A_4,\\ &\gamma^8(\tilde{\tau})=A_8-7A_7+15A_6-10A_5+A_4,\end{aligned}$$

etc.

Therefore

$$\mathrm{Dim}_\gamma(L^{2n+1}(p)\times L^{2m+1}(q)) = 2\sup\{k\mid A_k\neq 0\}.$$

COROLLARY 3.2([9]). $\mathrm{Dim}_\gamma(L^{2n+1}(p))$

$$=2\sup\{i\in N\mid i\leq n+1,\ \binom{n+1}{i}(\mathrm{re}(a))^i\neq 0\}.$$

THEOREM 3.3. *If* $\gamma^k(\tilde{\tau}_1)\neq 0$ *or* $\gamma^k(\tilde{\tau}_2)\neq 0$ *then* $\gamma^k(\tilde{\tau})\neq 0$, *where* $\tilde{\tau}_1=\tau(L^{2n+1}(p))-(2n+1)\in KO(L^{2n+1}(p))$ *and* $\tilde{\tau}_2=\tau(L^{2m+1}(q))-(2m+1)\in KO(L^{2m+1}(q))$.

Proof. Since $\gamma^k(\tilde{\tau}) = \gamma^k(\tilde{\tau}_1) + \gamma^k(\tilde{\tau}_2)$ + terms of the form $\binom{n+1}{i}\binom{m+1}{j}\{\prod_1^*\mathrm{re}(a)\}^i\{\prod_2^*\mathrm{re}(b)\}^j$ and

$$\prod{}_1^*\mathrm{re}(a) \in \prod{}_1^* KO(L^{2n+1}(p)), \quad \prod{}_2^*\mathrm{re}(b) \in \prod{}_2^* KO(L^{2m+1}(q)),$$
$$\{\prod{}_1^*\mathrm{re}(a)\}^i\{\prod{}_2^*\mathrm{re}(b)\}^j \in \prod{}_\wedge^* KO(L^{2n+1}(p) \wedge L^{2m+1}(q)),$$

this theorem comes from theorem (2.3).

Next we compute the γ-codimension of $L^{2n+1}(p) \times L^{2m+1}(q)$.

THEOREM 3.4. $\mathrm{Cod}_\gamma(L^{2n+1}(p) \times L^{2m+1}(q))$

$$=2\ \sup\{k \mid \binom{n+k}{k}(\prod{}_1^*\mathrm{re}(a))^k + \binom{m+k}{k}(\prod{}_2^*\mathrm{re}(b))^k + \sum_{\substack{i+j=k \\ i,j\geq 1}} \binom{n+i}{i}\binom{m+j}{j}(\prod{}_1^*\mathrm{re}(a))^i(\prod{}_2^*\mathrm{re}(b))^j \neq 0\}.$$

Proof. From the first part of the proof of theorem (3.1), we have $-\tilde{\tau} = -(n+1)\prod_1^*\mathrm{re}(a) - (m+1)\prod_2^*\ \mathrm{re}(b)$. Hence

$$\gamma_t(-\tilde{\tau}) = \{1 + \prod{}_1^*\mathrm{re}(a)t - \prod{}_1^*\mathrm{re}(a)t^2\}^{-(n+1)}\{1 + \prod{}_2^*\mathrm{re}(b)t - \prod{}_2^*\mathrm{re}(b)t^2\}^{-(m+1)}$$
$$= \sum_{i+j=0}^{\infty}(-1)^{i+j}\left\{\binom{n+i}{i}\binom{m+j}{j}(\prod{}_1^*\mathrm{re}(a))^i(\prod{}_2^*\mathrm{re}(b))^j(t-t^2)^{i+j}\right\}.$$

If we also set

$$B_k = \binom{n+k}{k}(\prod{}_1^*\mathrm{re}(a))^k + \binom{m+k}{k}(\prod{}_2^*\mathrm{re}(b))^k + \sum_{\substack{i+j=k \\ i,j\geq 1}} \binom{n+i}{k}\binom{m+j}{j}(\prod{}_1^*\mathrm{re}(a))^i(\prod{}_2^*\mathrm{re}(b))^j,$$

then, by taking the coefficient of t^i, we have

$$\gamma^0(-\tilde{\tau}) = 1, \qquad \gamma^1(-\tilde{\tau}) = -B_1, \qquad \gamma^2(-\tilde{\tau}) = B_2 + B_1,$$
$$\gamma^3(-\tilde{\tau}) = -B_3 - 2B_2, \qquad \gamma^4(-\tilde{\tau}) = B_4 + \binom{3}{1}B_3 + B_2,$$
$$\gamma^5(-\tilde{\tau}) = -B_5 - \binom{4}{1}B_4 - \binom{3}{2}B_3,$$
$$\gamma^6(-\tilde{\tau}) = B_6 + \binom{5}{1}B_5 + \binom{4}{2}B_4 - B_3,$$
$$\gamma^7(-\tilde{\tau}) = -B_7 - \binom{6}{1}B_6 - \binom{5}{2}B_5 - \binom{4}{3}B_4,$$
$$\gamma^8(-\tilde{\tau}) = B_8 + \binom{7}{1}B_7 + \binom{6}{2}B_6 + \binom{5}{3}B_5 + B_4,$$

etc.

Therefore

$$\mathrm{Cod}_\gamma(L^{2n+1}(p)\times L^{2m+1}(q))=2\sup\{k\,|\,B_k\neq 0\}.$$

COROLLARY 3.5([9]). $\mathrm{Cod}_\gamma(L^{2n+1}(p))$

$$=2\ \sup\{i\in N\,|\binom{n+i}{i}(\mathrm{re}(a))^i\neq 0\}.$$

THEOREM 3.6. *If* $\gamma^k(-\tilde{\tau}_1)\neq 0$ *or* $\gamma^k(-\tilde{\tau}_2)\neq 0$ *then* $\gamma^k(-\tilde{\tau})\neq 0$.

Proof. Since $\gamma^k(-\tilde{\tau})=\gamma^k(-\tilde{\tau}_1)+\gamma^k(-\tilde{\tau}_2)+$terms of the form $\binom{n+1}{i}\binom{m+j}{j}\{\prod_1{}^*\mathrm{re}(a)\}^i\{\prod_2{}^*\ \mathrm{re}(b)\}^j$ and

$$\prod{}_1{}^*\mathrm{re}(a)\in\prod{}_1{}^*\widetilde{KO}(L^{2n+1}(p)),\quad \prod{}_2{}^*\mathrm{re}(b)\in\prod{}_2{}^*\widetilde{KO}(L^{2m+1}(q)),$$
$$\{\prod{}_1{}^*\mathrm{re}(a)\}^i\{\prod{}_2{}^*\mathrm{re}(b)\}^j\in\prod{}_\wedge{}^*\widetilde{KO}(L^{2n+1}(p)\wedge L^{2m+1}(q)),$$

this theorem comes from theorem (2.3).

The order of $\mathrm{re}(a)^i$ in $\widetilde{KO}(L^{2n+1}(p))$ was computed by Kawaguchi–Sugawara.

THEOREM 3.7([5]). *For* $1\leqq i\leqq\left[\frac{n}{2}\right]$, *the element* $(\mathrm{re}(a))^i\in\widetilde{KO}(L^{2n+1}(p))$ *is of order* $p^{1+\left[\frac{n-2i}{p-1}\right]}$ *and* $(\mathrm{re}(a))^{\left[\frac{n}{2}\right]\pm 1}=0$, *where* $[y]$ *is the integral part of a real number* y.

For the next theorem, we set

$$k(n,p)=\max\left\{k\,|\,k\leqq\left[\frac{n}{2}\right],\ V_p\binom{n+1}{k}<1+\left[\frac{n-2k}{p-1}\right]\right\},$$

where $V_p(m)$ denote the p-adic valuation of m.

Let $\mathrm{Span}(M)$ denote the maximal number of linearly independent tangent vector fields over M.

THEOREM 3.8. $\mathrm{Span}(L^{2n+1}(p)\times L^{2m+1}(q))\leqq 2(n+m+1)-2\max\{k(n,p),\ k(m,q)\}$.

Proof. Let $k_0=2\max\{k(n,p),\ k(m,q)\}$. From the definition of $k(n,p)$, we have $0\leqq k_0\leqq 2(n+m+1)$. By theorem (3.3), (3.7) and corollary (3.2), we obtain $\gamma^{k_0}(\tau)\neq 0$. Applying theorem (2.1), we have Span $(L^{2n+1}(p)\times L^{2m+1}(q))\leqq 2(n+m+1)-k_0$.

For the next theorem, we set

$$l(n,p)=\max\left\{k\,|\,k\leqq\left[\frac{n}{2}\right],\ V_p\binom{n+k}{k}<1+\left[\frac{n-2k}{p-1}\right]\right\}.$$

As geometrical interpretation of theorem (3.6), we have nonembeddability and nonimmersibility of $L^{2n+1}(p)\times L^{2m+1}(q)$ into Euclidean spaces.

THEOREM 3.9. *(i)* $L^{2n+1}(p)\times L^{2m+1}(q)$ *cannot be immersible in* $R^{2(n+m+1)+2\max\{l(n,p),l(m,q)\}}$.

(ii) $L^{2n+1}(p)\times L^{2m+1}(q)$ *cannot be embeddable in* $R^{2(n+m+1)+2\max\{l(n,p),l(m,q)\}}$.

Proof. Let $l_0=2\max\{l(n,p), \ l(m,q)\}$. Using the definition of $l(n,p)$ and theorem (3.6), (3.7) corollary (3.5), we have $\gamma^{l_0}(-\tilde{\tau})\neq 0$. Applying Atiyah Criterion theorem (2.2), we can get the desired results.

References

1. M.F. Atiyah, *Immersions and embeddings of manifolds*, Topology **1**(1962), 125-132.
2. ______, *K-theory*, Benjamin Inc. New York (1967).
3. D. Husemoller, *Fibre bundle*, McGraw-Hill New York (1966).
4. T. Kambe, *The structure of K_Λ-rings of the lens space and their applications*, J. Math. Soc. Japan **18**(1966), 135-146.
5. T. Kawaguchi and M. Sugawara, *K-and KO-rings of the lens space $L^n(p)$ for odd prime p*, Hiroshima Math. J. **1**(1971), 273-286.
6. T. Kobayashi and M. Sugawara, *K_Λ-rings of lens space $L^n(4)$*, Hiroshima Math. J. **1**(1971), 253-271.
7. J. Milnor and J. Stasheff, *Characteristic classes*, Annals of Math. Studies **76**, Princeton Univ. Press, Jersey (1974).
8. H. Suzuki, *Operations in KO-theory and products of real projective spaces*, Mem. Fac. Sci. Kyushu Univ. Ser. A **18**(1964), 140-153.
9. M. Yasuo, *γ-dimension and products of lens spaces*, Mem. Fac. Sci. Kyushu Univ. Ser. A **31**(1977), 113-126.

Kyungpook University
Taegu 635, Korea