韓國數學敎育學會誌〈數學敎育〉
1987. 5, Vol. XXV, No. 2

Journal of the Korea Society of Mathematical Education
May. 1987, Vol. XXV, No. 2

The Krull dimension with an application of the valuative dimension

by Park, Chan-Bong

Won-Kwang University

1. Introduction

Let $R^{(k)} = R[x_1, x_2, \ldots, x_k]$. be polynomial ring in k indeterminates over a commutative ring R with identity and let dim $R^{(k)}$ denote the Krull dimension of $R^{(k)}$ for each non-negative integer k.

In [5] and [6], A. Seidenberg proved that if dim $R=1$, dim $R[x]=2$ if and only if the integral closure of R, $\overline{R}$, is a Prüfer domain.

Finally he proved that if R is a Prüfer domain then

dim $R^{(k)} =$ dim $R+k$ for positive integer k.

In [1], P. Jaffard introduced the notion of valuative dimension $dim_v R$, and he also proved that

$dim_v R = Sup\{dim\ v \mid v$ is a valuation overring of $R\}$.

In [2], he also established the properties of $dim_v R$:

a) $dim\ R \leqq dim_v R$.

b) $dim_v R = dim\ R \leftrightarrow dim\ R^{(k)} = k + dim\ R$ for all non-negative integer k.

In this note we will discuss the followings:

$*_1$ What would be ring which the Krull dimension coincides with the valuative dimension?

$*_2$ What would be ring such that $dim\ R^{(k)} = dim\ R+k$ for all non-negative integer k?

$*_2$ is a result of Krull & Seidenberg that we have already cited.

But here the same results of W. Krull & A. Seidenberg and the answer of $*_1$ will be given by using the property of the valuative dimension. First we will list the lemma well-known without the proof[3].

1) Any invertible ideal in a quasi-local domain is principal.

2) Let I be an invertible ideal in an integral domain R, and let S be a multiplicative closed set in R. Then I_S is invertible in R_S.

3) Let I be a finitely generated ideal in an integral domain R. Then I is invertible if and only if I_M is Principal for every maximal ideal M.

4) A quasi-local domain is a valuation domain if and only if it is a Bézout domain.

Proposition 1. *T. F. A. E. for an integral domain R* [3]:

1) *R is prüfer;*

2) *For every prime ideal P, R_P is a valuation domain;*

3) *For every maximal ideal M, R_M is a valuation domain.*

Proof. 1)→2). Let J be a finitely generated non-zero ideal in R_P. If J is generated by $a_1/s_1, \dots, a_n/s_n (a_i, s_i \in R,\ s_i \notin P)$, then $J = J_p$, where $I = (a_1, \dots, a_n)$. By hypothesis I is invertible; hence by Lemma 1) & 2) J is principal. By Lemma 4) R_p *is a valuation domain.*

2) → 3). Trivial.

3) → 1). Let I be an non-zero finitely generated ideal in R. Then every I_M is principal, so by Lemma 3) I is invertible.

Proposition 2. *Let R be a prüfer domain with quotient field K, and let V be a valuation domain between R and K. Then $V = R_P$ for some prime ideal P in R* [3].

Proof. Let M be the unique maximal ideal of V and set $P = M \cap R$. For any s in R but not in P we must have $s^{-1} \in V$, for otherwise $s \in M$ and so $s \in P$. Thus $R_P \subset V$.

To prove that $V \subset R_P$ we note (Proposition 1) that R_P is a valuation domain. So if we take $v \in V$ and find $v \notin R_P$ we must have $v^{-1} \in R^P$, say $v^{-1} = a/s,\ a, s \in R,\ s \notin P$. Here $a \in P$ for otherwise a/s would be unit in R_P and $v \in R_P$, which we assumed is not the case. Hence $a \in M$ and $av \in M$, $s = av \in M \cap R = P$, again a contradiction.

Theorem 1. *If R is a Prüfer domain,*

$$dim\ R^{(n)} = dim\ R + n \text{ for all } n.$$

Proof. To prove this it is enough to show that if R is a Prüfer domain then $dim\ R = dim_v R$ because of property b) of the valuative dimension. By proposition 2 $V = R_P$ for some $P \in$ space (R).

Hence $dim_v R = sup\{dim\ V \mid V \text{ are valuation overrings of } R\}$

$= sup\{dim\ R_P \mid P \in spec\ (R)\}$ spec

$= sup\{dim\ P \mid P \in \Omega(R),\ maximal\ spectrum\}$

$= dim\ R.$

Theorem 2. *If R is a Noether domain, then $dim_v R = dim\ R$.*

Proof. Since $dim\ R^{(n)} = dim\ R + n$ for all n if R is noetherian, by property b) it follows that $dim_v R = dim\ R$.

References

1. Jaffard. P., "*Dimension des anneaux de polynomes: La notion de dimension valuative*", C.R. Acad. Sci. Paric ser A-B, 246(1958), 3305-3307.
2. Jaffard, Paul, "*Theorie de la dimension dans les Anneaux de polynomes*", Gauthier-Villars, (1960).
3. Kaplanski, I., *Commutative rings,* Univ. of Chicago, (1974).
4. Krull, W., "Jacobsconsche rings, Hilbertscher Nullstellensatz, Dimensionentheorie", *Math. Z.* 54 (1951), 354-387.

5. Seidenberg, A., "A note on the dimension theory of rings", *Pacific J. Math.* 3 (1953), 505-512.
6. Seidenberg, A., "On the dimension theory of rings, II", *Pacific J. Math.* 4(1954), 603-614.