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Note On the (Cr)-space and the Closed Graph Theorem

by Ha Yeoung-Soon, Son Yong-Kyu, Lee Sang-Kyu and Lau Jeung-Hark

Chin Ju Teachers College, Chin Ju, Korea

— Dedicated to Professor Han Shick Park on his 60th birthday —

1. Introduction

It is known that the closed graph theorem holds if E is barrelled and F is (Br)-complete [5, 7]. 'ollowing the work of MacIntosh [3], we define a class of spaces (called (Cr)-spaces) which ; larger than the class of (Br)-complete spaces, and show that the closed graph theorem still olds if E is barrelled and F a (Cr)-space. Also we show that the open mapping theorem holds E is a (C)-space and F barrelled, and that if the closed graph theorem holds for mappings of ıy barrelled space E into F, the space F is the (Cr)-space.

We use the notation and terminology of Wilansky [8]. By a LCTVS, we mean a (real or comlex) locally convex topological vector space, always assumed separated. If $[E, F]$ is a dual pair, ıen we denote the weak topology on E by $\sigma(E, F)$. In particular, if E is a LCTVS and E' its pological dual, then the associated weak topology on E is denoted by $\sigma(E, E')$. The topology (E', E) on E' is called the weak (rather than weak*) topology. Now let T be a linear mapping om a LCTVS E to another F, then we call the linear mapping T' from F' into E' adjoint of .e linear mapping T if $T'(F') \subset E'$. A LCTVS E is called ultra bornological if it can be reprented as the locally convex hull $\sum_{\alpha} A_\alpha(E_\alpha)$ of Banach spaces E_α [2]. An ultra bornological space is ›th bornological and barrelled. Conversely, every sequentially complete bornological space is ultra›rnological, all topologies mentioned are assumed to be locally convex. Let E' be the dual of a CTVS E. E_σ' will denote the point set E' endowed with the topology $\sigma(E', E)$.

2. (Cr)-spaces

By analogy with the (B)-complete and (Br)-complete spaces, we define (C)-and (Cr)-spaces. ıe following definition is due to MacIntosh [3].

Definition. A LCTVS E is a (C)-*space* (resp. (Cr)-*space*) if every linear subspace (resp. dense .ear subspace) D of F_σ' whose intersection with each weakly bounded subset B of F' is weakly ›sed in B, is necessarily closed in F_σ'.

It is clear that every (C)-space is a (Cr)-space. Also, every (B)-complete (resp. (Br)-complete) ıce is a (C)-space (resp. (Cr)-space). In the definition we could require that the set B be ıvex.

Proposition 2-1. *Every barrelled* (C)*-space* (resp. (Cr)*-space*) *is* (B)*-complete* (resp. (Br)*-complete*).

Proof. The properties of being equicontinuous and weakly bounded are equivalent for any subset of the dual of the barrelled space.

Proposition 2-2. *Let E be a closed subspace of the LCTVS F. If F is a* (C)*-space or a* (Cr)*-space, then E (under the induced topology) has the same property.*

Proof. This can be proved by adapting the proof for (B)-complete and (Br)-complete spaces in [7].

We shall need the following property of (C)-spaces.

Lemma 2-3. *If E is a closed subspace of a (C)-space F, then F/E, in its quotient topology, is a (C)-space.*

Proof. Let $t: E \longrightarrow F$ be a quotient map (=linear, continuous, open) where E is a (C)-space Let S be a subspace of F_σ' whose intersection with each weakly subset B of F' is weakly closed in B. Then t' preserves the property of S. Then $t'(S)$ is weakly closed. Thus $(t')^{-1}[t'(S)]$ is weakly closed since t' is weakly continuous. This set is S since t' is one-to-one.

3. The closed graph theorem

A linear mapping T from a LCTVS E into another F such that T has closed graph is called simply a *closed graph liner mapping*.

Proposition 3-1. *If T is a closed graph linear mapping of a barrelled space E into a (Cr)-space F, then T is continuous.*

Proof. Since T is closed, the domain $D(T')$ of T' is weakly dense in F', and T' is continuous if $D(T')$ is provided with the topology induced by $\sigma(F', F)$ and E' with $\sigma(E', E)$. We claim that the proposition follows if we can prove that $D(T') \cap B$ is weakly closed for every weakly bounded subset B of F'. Indeed, since F is a (Cr)-space, we then conclude that $D(T') = F'$. Therefore T' maps weakly relatively compact, and in particular equicontinuous subsets of F' into weakly relatively compact subsets of E'. But $\sigma(E', E)$-compact subsets of E' are equicontinuous and hence it follows that, given a neighborhood V of zero in F, there exists a neighborhood U of zero in E such that $T(U) \subset V$, that is, T is continuous.

To prove that $D(T') \cap B$ is weakly closed, let y' belong to the weakly closure of $D(T') \cap B$ and let y_α' be a net in $D(T') \cap B$ converging weakly to y'. Then, by the weakly continuity of T' its image $T'y_\alpha'$ is a bounded Cauchy net for $\sigma(E', E)$. But weakly bounded subsets of E' are weakly relatively compact, and hence $T'y_\alpha'$ converges to some point x' of E'. Since T' is weakly closed in $F' \times E'$, this proves that $y' \in D(T') \cap B$ and hence $D(T') \cap B$ is weakly closed. The proof of the proposition is complete.

Corollary 3-2. *Suppose that E is ultra bornological and that F is a* (Cr)*-space. If T is a closed graph linear mapping of E into F, then T is continuous.*

The following is the partial converse of the previous proposition.

Proposition 3-3. *Suppose that F satisfies the following requirement: There is a weakly dense ubspace G of F such that*

(a) *$B \subset F'$ is weakly bounded if and only if B is $\sigma(F', G)$-bounded,*

(b) *weakly bounded subsets of F' are relatively compact for $\sigma(F', G)$. Then, if the closed graph theorem holds for mappings of any barrelled space E into F, the space F is a (Cr)-space.*

Proof. Let L be a weakly dense subspace of F' such that $L \cap B$ is weakly closed for weakly ounded subset B of F'. Consider G in the topology of uniform convergence on sets of the form $\cap B$. It follows from (a) and (b) that G is barrelled and $G' = L$. If T is the natural injection of G into F, then clearly $D(T') = L$. Since T is continuous and hence closed in the topologies (G, L) and $\sigma(F, L)$, we conclude that T is closed also in the stronger original topologies of G nd F. Consequently T is continuous, and in particular $D(T') = F'$. Hence $L = F'$, which completes he proof.

There are two kinds of open mapping theorems. If T is a linear mapping of E into F, we may ppose either that T has closed graph or that T is continuous, and conclude, with suitable hyotheses on E and F, that T is open. Here we deal with the latter.

Proposition 3-4. *Let E and F be two LCTVSs. Suppose that E is a (C)-space and F barelled, and that T is a continuous linear mapping of E into F. Then T is open.*

Proof. Since T is continuous, $T^{-1}(0)$ is closed. Let $H = E/T^{-1}(0)$. Then H is a (C)-space by emma 2-3. Then we can write $T = s \circ \phi$ where ϕ is the canonical mapping of E onto H and s a one-to-one continuous linear mapping of H onto F. The graph of s^{-1}, which the same as the raph of s, is therefore closed in $F \times H$, and so s^{-1} is continuous. Hence T is open.

We call the linear mapping T of the LCTVS E into the LCTVS F *nearly open* if $\overline{T(U)}$ is a eighborhood in F for every neighborhood U in E, and nearly continuous if $\overline{T^{-1}(V)}$ is a neighborhood in E for every neighborhood V in F. When T is one-to-one and onto, clearly T is nearly ntinuous if and only if its inverse is nearly open.

Proposition 3-5. *Let T be a continuous and nearly open linear mapping of a LCTVS E onto LCTVS F. If E is a (C)-space, then so is F.*

Proof. Let D be a subspace of F_σ' whose intersection with each weakly bounded subset B of F' is weakly closed in B. Let us examine the subspace $T'(D)$ in E'. We have

$T'(D) \cap U^\circ = T'([T(U)]^\circ \cap D)$ for every neighborhood U of zero in E where U° is the polar U.

Now $[T(U)]^\circ$ is the polar of a neighborhood of zero in F, so that $[T(U)]^\circ \cap D$ is compact in $F', F)$. Hence $T'(D) \cap U^\circ$ is weakly compact for every U so that $T'(D)$ is closed in E'. It llows that D is closed in F'. Hence F is a (C)-space.

Proposition 3-6. *Let T be a one-to-one linear mapping of a LCTVS E onto a closed subspace a LCTVS F. Suppose that T is open and nearly continuous. Then E is a (C)-space if F is.*

Proof. Let us denote by F_1, the space $T(E)$. According to our assumption F_1 is closed in F so that F_1 is a (C)-space. Let us denote by g the linear mapping from F_1 onto E which is inverse to T.

Clearly g is continuous and nearly open. It follows that g is open so that T is both open and continuous. Hence E is both algebraically and topologically isomorphic to the (C)-space F_1.

Thus E is a (C)-space.

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