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Positive Linear Functionals and Representations

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——Dedicated to Professor Han Shick Park on his 60th birthday——

1. Introduction

We assume throughout that $\mathfrak{A}$ is a C^*-algebra, $\mathscr{B}$ is a closed *-subalgebra, $\mathscr{A}$ is an involutive Banach algebra, and $\mathfrak{A}_h$ is a real vector space consisting of all self-adjoint elements of $\mathfrak{A}$. Let $\mathscr{S}$ be a subset of $\mathfrak{A}$. We say that a net $(V_\lambda, \lambda\in\Lambda, \geq)=\{V_\lambda\}$ of self-adjoint element of $\mathscr{S}$ is an increasing approximate identity for $\mathscr{S}$ if $\|S-SV_\lambda\|_\lambda\to 0$, $\|S-V_\lambda S\|_\lambda\to 0$ for each S in $\mathscr{S}$, while $0\leq V_\lambda\leq V_\mu\leq I$ whenever $\lambda, \mu\in\Lambda$ and $\lambda\leq\mu$ where $\Lambda=\{B\in\mathscr{B}\cap\mathfrak{A}^+ : \|B\|<1\}$.

In this note, we first show that, if φ is a positive linear functional on $\mathscr{A}$ with a bounded approximate identity, then two facts that φ is the pure and that the cyclic representation induced by φ is irreducible, are equivalent.

The main result in this note is to show that every A^*-algebra admit sufficiently many irreducible representations. In particular, every C^*-algebra admits sufficiently many irreducible representations.

2. Preliminary

By a ***representation*** of C^*-algebra $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$, we mean a *-homomorphism φ from $\mathfrak{A}$ into C^*-algebra $B(\mathcal{H})$ of all bounded operators on $\mathcal{H}$. The Hilbert space $\mathcal{H}$ is called the representation space of φ. If, in addition, φ is one-to-one (hence, a *-isomorphism) it is described as a faithful representation.

Suppose that φ is a representation of $\mathfrak{A}$ on $\mathcal{H}$, $\varphi(I)=1$. $\|\varphi(A)\|\leq\|A\|$ for each $A\in\mathfrak{A}$ (whence φ is continuous) and $\|\varphi(A)\|=\|A\|$ if φ is faithful. The set $\{A\in\mathfrak{A} : \varphi(A)=0\}$ is a closed two-sided ideal in $\mathfrak{A}$, the kernel of φ.

If there is a vector $\xi\in\mathcal{H}$ for which the linear subspace $\varphi(\mathfrak{A})\xi=\{\varphi(A)\xi : A\in\mathfrak{A}\}$ is everywhere dense in $\mathcal{H}$, φ is described as a *cyclic* representation, and ξ is termed a *cyclic vector* for φ.

In order to specify the representation space together with a representation, we write $\{\Pi, \mathcal{H}\}$ or $\mathcal{H}_\pi$. Two representations $\{\Pi_1, \mathcal{H}_1\}$ and $\{\Pi_2, \mathcal{H}_2\}$ of $\mathfrak{A}$ are said to *unitary equivalent* if there exists an isometry U of $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that $U\Pi_1(A)U^*=\Pi_2(A)$, $A\in\mathscr{A}$: we write this fact as $\{\Pi_1, \mathcal{H}_1\}\cong\{\Pi_2, \mathcal{H}_2\}$ or $\Pi_1\cong\Pi_2$. If $\Pi(A)\neq 0$ for every nonzero $A\in\mathscr{A}$, then Π is called faithful. Let $\{\Pi, \mathcal{H}\}$ be a representation of $\mathscr{A}$. For any pair ξ, η in $\mathcal{H}$, we define a functional $\rho(\Pi : \xi, \eta)$ by

$$\langle A, \rho(\Pi : \xi, \eta)\rangle=(\Pi(A)\xi|\eta), \quad A\in\mathscr{A}.$$

It is then obvious that

$$\rho(\Pi : \xi, \eta)^* = \rho(\Pi : \eta, \xi)$$
$$\rho(\Pi : \xi, \xi) \geqq 0.$$

The positive linear functional $\rho(\Pi : \xi, \xi)$ is often abbreviated as $\rho(\Pi : \xi)$.

Definition 2.1. A representation $\{\Pi, \mathcal{H}\}$ of an involutive Banach algebra $\mathcal{A}$ is said to be ***proper*** or ***nondegenerate*** if for any nonzero $\xi \in \mathcal{H}$, there exists an element $A \in \mathcal{A}$ with $\Pi(A)\xi \neq 0$. Otherwise, the closed subspace $[\Pi(A)\mathcal{H}]$ is called the ***essential space*** of Π and denoted by $\mathcal{H}(\Pi)$.

We mainly consider nondegenerate representation, so we mean by a representation a nondegenerate one unless there is danger of confusion.

Definition 2.2. A linear functional ρ on an involutive Banach algebra $\mathcal{A}$ is called ***positive*** if $\rho(A^*A) \geqq 0$ for every $A \in \mathcal{A}$. A positive linear functional of norm one is called a ***state.*** If $\rho(A^*A) \neq 0$ for every nonzero $A \in \mathcal{A}$, then ρ is said to be ***faithful.***

Lemma 2.3. *If ρ is a positive linear functional on $\mathcal{A}$,* then

$$\rho(B^*A) = \overline{\rho(A^*B)} \qquad (1)$$
$$|\rho(B^*A)|^2 \leq \rho(A^*A)\rho(B^*B), \quad A, B \in \mathcal{A}. \qquad (2)$$

Inequality (2) is called *the Cauchy-Schwartz inequality.* If $\mathcal{A}$ is unital, then (1) and (2) imply the following

$$\rho(A^*) = \overline{\rho(A)} \qquad (3)$$
$$|\rho(A)|^2 \leq \rho(I)\rho(A^*A), \quad A \in \mathcal{A}. \qquad (4)$$

Lemma 2.4. *If ρ is a positive linear functional, then $N_\rho = \{A \in \mathcal{A} : \rho(A^*A) = 0\}$ is a left ideal.*

Definition 2.5. The left ideal N_ρ is called *the left kernel of ρ.* Similarly, we can define the right kernel of ρ.

Suppose now a positive linear functional on $\mathcal{A}$ is given. For any $A \in \mathcal{A}$, let $\eta_\rho(A)$ denote the coset $A + N_\rho$ in the quotient space $\mathcal{A}/N_\rho$. We equip the complex vector space $\mathcal{A}/N_\rho$ with the inner product defined by

$$(\eta_\rho(A) \mid \eta_\rho(B)) = \rho(B^*A), \quad A, B \in \mathcal{A} \qquad (5)$$

We denote by $\mathcal{H}_\rho$ the Hilbert space obtained as the completion of $\mathcal{A}/N_\rho$. We shall see soon tha the linear operator $\eta_\rho(A) \in \mathcal{A} | N_\rho \rightarrow \eta_\rho(BA) \in \mathcal{A}/N_\rho$ for each $B \in \mathcal{A}$ is extended to a bounded operato $\Pi_\rho(B)$ on the Hilbert space $\mathcal{H}_\rho$, and the map, $\Pi_\rho : B \in \mathcal{A} \rightarrow \Pi_\rho(B) \in \mathcal{L}(\mathcal{H}_\rho)$ is indeed a representatio of $\mathcal{A}$. To do this, we need a few lemmas.

3. Technical Lemmas

Lemma 3.1. *If A is an element of $\mathcal{A}$ with $\|1 - A\|_{s,\rho} < 1$, then there exists $B \in \mathcal{A}$* with $B^2 = A$ *Furthermore, if A is hermitian then a selfadjoint element can be chosen as the above B.*

Lemma 3.2. *If $\mathcal{A}$ is a unital involutive Banach algebra then every positive linear functiona ρ of $\mathcal{A}$ is continuous and $\|\rho\| = \rho(1)$.*

Lemma 3.3. *Let ρ be a positive linear functional on A. For each $A \in \mathcal{A}$, we set $\rho_A(B) = \rho(ABA^*)$*

$B\in\mathscr{A}$. Then ρ_A is a continuous positive linear functional and $\|\rho_A\|\leq\rho(AA^*)$.

Lemma 3.4. *Let $\mathscr{A}$ be an involutive Banach algebra with a bounded approximate identity $\{u_i\}$ of norm$\leq\gamma$. If ρ is a continuous positive linear functional on $\mathscr{A}$, then*

i) $\rho(A^*)=\overline{\rho(A)}$,

ii) $|\rho(A)|^2\leq\gamma^2\|\rho\|\rho(A^*A)$, $A\in\mathscr{A}$.

Lemma 3.5. *A positive linear functional C^*-algebra is continuous.*

Lemma 3.6. *Let $\mathscr{A}$ be an involutive Banach algebra with a bounded approximate identity. To any (continuous) positive linear functional ρ, there corresponds uniquely, within unitary equivalence, a representation $\{\Pi_\rho,\mathscr{H}_\rho\}$ of $\mathscr{A}$ with a vector ξ_ρ such that*

i) $[\Pi_\rho(\mathscr{A})\xi_\rho]=\mathscr{H}_\rho$.

ii) $\rho(A)=(\Pi_\rho(A)\xi_\rho|\xi_\rho)$, $A\in\mathscr{A}$.

Definition 3.7. The representation $\{\Pi_\rho,\mathscr{H}_\rho\}$ constructed in the above lemma is called *the cyclic representation of $\mathscr{A}$ by ρ.* Sometimes it is denoted by $\{\Pi_\rho,\mathscr{H}_\rho,\xi_\rho\}$ to indicate the vector corresponding to ρ. The construction of $\{\Pi_\rho,\mathscr{H}_\rho,\xi_\rho\}$ employed above is called *the Gelfand-Naimark-Segal Construction.* In general, if a representation $\{\Pi,\mathscr{H}\}$ of $\mathscr{A}$ admits a vector ξ such that $[\Pi(\mathscr{A})\xi]=\mathscr{H}$ where $[\mathscr{M}]$ denotes the closed subspace of $\mathscr{H}$ spanned by $\mathscr{M}$ for any subset $\mathscr{M}$ of $\mathscr{H}$, then $\{\Pi,\mathscr{H}\}$ is said to be *cyclic* and ξ is called *a cyclic vector* for Π.

Let $\{\{\Pi_i,\mathscr{H}_i\}:i\in I\}$ be a family of representation of $\mathscr{A}$. Let $\mathscr{H}$ be the direct sum Hilbert space $\sum_{i\in I}\oplus\mathscr{H}_i$.

For each vector $\xi=\sum_{i\in I}\oplus\xi_i$, $\xi_i\in\mathscr{H}_i$ and $A\in\mathscr{A}$, put $\Pi(A)\xi=\sum_{i\in I}\oplus\Pi_i(A)\xi_i$, $\Pi(A)\xi$ is a vector of $\mathscr{H}$, $\Pi(A)$ is a bounded operator on $\mathscr{H}$. The representation $\{\Pi,\mathscr{H}\}$ is called the *direct sum of* $\{\{\Pi_i,\mathscr{H}_i\}:i\in I\}$ and denoted $\sum_{i\in I}\oplus\{\Pi_i,\mathscr{H}_i\}$. Each $\{\Pi_i,\mathscr{H}_i\}$ is called a *component of* $\{\Pi,\mathscr{H}\}$.

Definition 3.8. Given a representation $\{\Pi,\mathscr{H}\}$ of $\mathscr{A}$, a closed subspace $\mathscr{M}$ of $\mathscr{H}$ is called *an invariant subspace* of $\{\Pi,\mathscr{H}\}$ if $\Pi(A)\mathscr{M}\subset\mathscr{M}$ for each $A\in\mathscr{A}$. In this case, the restriction $\Pi(A)|\mathscr{M}$ of $\Pi(A)$ to $\mathscr{M}$ give rise to a new representation of $\mathscr{A}$ on, $\mathscr{M}$ which will be denoted by Π_m and called a *subrepresentation of* Π.

It is routine to show that the orthogonal complement $\mathscr{M}^\perp$ of any invariant subspace $\mathscr{M}$ of $\{\Pi,\mathscr{H}\}$ is also invariant and that $\{\Pi,\mathscr{H}\}\cong\{\Pi_m,\mathscr{M}\}\oplus\{\Pi_{m'},\mathscr{M}^\perp\}$. If $\{\Pi,\mathscr{H}\}$ has no invariant subspace other than $\mathscr{H}$ and $\{0\}$, then it is said to be *irreducible.*

Lemma 3.9. *Every nondegenerate representation of* $\{\Pi,\mathscr{H}\}$ *of an involutive Banach algebra $\mathscr{A}$ is a direct sum of cyclic representations.*

Lemma 3.10. *A C^*-algebra admits a faithful representation. Hence it is isometrically isomorphism to a uniformly closed selfadjoint algebra of operators on $\mathscr{H}$.*

Proof. Let $\mathfrak{A}$ be a C^*-algebra. If A is a nonzero element of $\mathfrak{A}$, then $-A^*\mathscr{A}\in\mathfrak{A}^+$. Since $\mathfrak{A}^+$ is closed convex cone in the real Banach space $\mathfrak{A}_h$, there exists, by the Hahn-Banach theorem, a linear functional f_A on $\mathfrak{A}_h$ such that $f_A(B)\geqq0$, $\forall B\in\mathfrak{A}^+$ and $f_A(-A^*\mathscr{A})<0$. We extend f_A to the whole algebra $\mathfrak{A}$ as follow:

$$f_A(B+iC)=f_A(B)+if_A(C),\quad B,C\in\mathfrak{A}_h.$$

If follows that f_A is a positive linear functional with $f_A(A^*A)>0$. Let $\{\Pi_A, \mathcal{H}_A, \xi_A\}$ be the cyclic representation of $\mathfrak{A}$ induced by f_A. We then have

$$\|\Pi_A(A)\xi_A\|^2=(\Pi_A(A^*A)\xi_A|\xi_A)=f_A(A^*A)>0$$

so that $\Pi_A(A)\neq 0$. Put $\{\Pi, \mathcal{H}\}=\sum_{A\in\sigma-\{0\}}\oplus\{\Pi_A, \mathcal{H}_A\}$. It follows that Π is faithful. Hence Π is a *-isomorphism of $\mathfrak{A}$ onto $\Pi(\mathfrak{A})$, therefore it is an isometry.

Definition 3. 11. An involutive Banach algebra is called an A^**-algebra* if it admits a faithful representation. Of course, a C^*-algebra is an A^*-algebra just shown. Let $\mathcal{B}$ be an A^*-algebra. We define a new norm $\|\cdot\|_*$ in $\mathcal{B}$ by $\|A\|_*=sup\{\|\Pi(A)\|: \Pi$ runs over all representations of $\mathcal{B}\}$ for $A\in\mathcal{B}$. Then $\|A\|_*\leq\|A\|$ for $A\in\mathcal{B}$. The completion of $\{\mathcal{B}, \|\cdot\|_*\}$ is a C^*-algebra which will be called the ***enveloping*** C^*-algebra of $\mathcal{B}$ and denote $C^*(\mathcal{B})$.

Lemma 3. 12. *If* $\{\Pi, \mathcal{H}\}$ *is a representation of* $\mathcal{A}$, *then the following two conditions are equivalent.*

i) $\{\Pi, \mathcal{H}\}$ *is irreducible,*

ii) *Only scalar multiplication operators commute with* $\Pi(\mathcal{A})$.

Definition 3. 13. A positive linear functional φ of $\mathcal{A}$ is called ***pure*** if every positive linear functional Ψ on $\mathcal{A}$, majorized by φ in the sense that $\Psi(\mathcal{A}_*A)\leq\varphi(A^*\mathcal{A})$, is of the form $\lambda\varphi$ $0\leq\lambda\leq 1$. We denote the set of all pure states by $\mathcal{P}(\mathcal{A})$.

4. Main results

The main result of this note is devoted to the existence of irreducible representations and the characterization in terms of positive linear functional. We will state and prove after a lemma.

Lemma 4. 1. *If* φ *is a (continuous) positive linear functional on* $\mathcal{A}$ *with a bounded approximate identity, then the following two statements are equivalent:*

i) φ *is pure,*

ii) *The cyclic representation* $\{\Pi_\varphi, \mathcal{H}_\varphi, \xi_\omega\}$ *induced by* φ *is irreducible*

Proof. i)$\Rightarrow$ii) : Let R be an invariant closed subspace of $\mathcal{H}$ and P be the projection of $\mathcal{H}$ on R. It follows that P commutes with every $\Pi_\varphi(A)$, $A\in\mathcal{A}$. Putting

$$\rho(A)=(\Pi_\varphi(A)\rho\xi_\varphi|\rho\xi_\varphi),\quad A\in\mathcal{A}.$$

We obtain a continuous positive linear functional ρ on $\mathcal{A}$. Since we have

$$\rho(A^*A)=\|\Pi_\varphi(A)P\xi_\omega\|^2=\|P\Pi_\varphi(A)\xi_\varphi\|^2\leq\|\Pi_\varphi(A)\xi_\varphi\|^2=\varphi(A^*A),\quad A\in\mathcal{A}.$$

ρ is majorized by φ, so that $\rho=\lambda\varphi$, $0\leq\lambda\leq 1$, by assumption.

Hence we have, for every $A, B\in\mathcal{A}$,

$$\begin{aligned}(\lambda\Pi_\varphi(A)\xi_\varphi|\Pi_\varphi(B)\xi_\varphi)&=\lambda_\varphi(B^*A)=\rho(B^*A)\\&=(\Pi_\varphi(A)P\xi_\varphi|\Pi_\varphi(B)P\xi_\varphi)=(P\Pi_\varphi(A)\xi_\varphi|P\Pi_\varphi(B)\xi_\varphi)\\&=(P\Pi_\varphi(A)\xi_\varphi|\Pi_\varphi(B)\xi_\varphi\end{aligned}$$

which implies that $P=\lambda 1$. Thus $P=0$ or 1. Therefore Π_φ is irreducible.

ii)$\Rightarrow$i) : Suppose ρ is a continuous positive linear functional on $\mathcal{A}$ majorized by φ. On the dense subsapce $\Pi_\varphi(\mathcal{A})\xi_\varphi$ of $\mathcal{H}_\varphi$, define a new inner product by

$$<\Pi_\varphi(A)\xi_\varphi|\Pi_\varphi(B)\xi_\varphi>=\rho(B^*A),\quad A, B\in\mathcal{A}.$$

It follows that the new inner product is majorized by the original one in $\mathcal{H}$, so that the new inner product makes sense, and there exists a bounded positive operator L of norm≤ 1 on $\mathcal{H}$, such that

$$\langle \xi | \eta \rangle = (L\xi | \eta), \quad \xi, \eta \in H.$$

For every $A, B, C \in \mathcal{A}$, we have

$$\begin{aligned}(L\Pi_\varphi(A)\Pi_\varphi(B)\xi_\varphi | \Pi_\varphi(C)\xi_\varphi) &= \langle \Pi_\varphi(A)\Pi_\varphi(B)\xi_\varphi | \Pi_\varphi(C)\xi_\varphi \rangle \\ &= \rho(C^*AB) = \rho((A^*C)^*B) \\ &= \langle \Pi_\varphi(B)\xi_\varphi | \Pi_\varphi(A^*)\Pi_\varphi(C)\xi_\varphi \rangle \\ &= (L\Pi_\varphi(B)\xi_\varphi | \Pi_\varphi(A)^*\Pi_\varphi(C)\xi_\varphi) \\ &= (\Pi_\varphi(A)L\Pi_\varphi(B)\xi_\varphi | \Pi_\varphi(C)\xi_\varphi).\end{aligned}$$

Hence L commutes with $\Pi_\varphi(A)$, $A \in \mathcal{A}$. By Lemma 3.12, L must be of the form $\lambda 1$. Therefore, we get

$$\begin{aligned}\rho(B^*A) &= (L\Pi_\varphi(A)\xi_\varphi | \Pi_\varphi(A)\xi_\varphi) \\ &= \lambda(\Pi_\varphi(A)\xi_\varphi | \Pi_\varphi(B)\xi_\varphi) = \lambda\varphi(B^*A), \quad A, B \in \mathcal{A}.\end{aligned}$$

Since the set of all B^*A is dense in $\mathcal{A}$ by the existence of an approximate identity, we have $\rho = \lambda\varphi$. The inequality $0 \leq \lambda \leq 1$ follows from the fact that $0 \leq L \leq 1$.

Theorem 4.2. *An A^*-algebra $\mathcal{A}$ admits sufficiently many irreducible representations, i.e, for any nonzero $A \in \mathcal{A}$, there exists an irreducible representation Π of $\mathcal{A}$ with $\Pi(A) \neq 0$. In particular, every C^*-algebra admits sufficiently many irreducible representations.*

Proof. Let $\mathcal{B}$ denote the enveloping C^*-algebra $C^*(\mathcal{A})$ of $\mathcal{A}$. Since $\mathcal{A}$ is dense in $\mathcal{B}$, the restriction of any irreducible representation of $\mathcal{B}$ of $\mathcal{A}$ is irreducible. Therefore, it suffices to prove the existence of sufficiently many irreducible representation of $\mathcal{B}$. Let $\mathcal{C}$ denote the set of all positive linear functionals on $\mathcal{B}$ of norm≤ 1. It follows that $\mathcal{C}$ is a $\sigma(\mathcal{B}^*, \mathcal{B})$-compact convex subset of the conjugate space $\mathcal{B}^*$ of $\mathcal{B}$ as a Banach space. Extreme point of $\mathcal{C}$ are either pure states or zero, so that the Krein-Milman theorem says that $\mathcal{C}$ is the $\sigma(\mathcal{B}^*, \mathcal{B})$-closed convex closure of zero and pure states. Therefore, if $\rho(A) = 0$, $A \in \mathcal{B}$ for every pure state ρ of $\mathcal{B}$, then $\varphi(A) = 0$ for every $\varphi \in \mathcal{C}$; hence $(\Pi(A)\xi | \xi) = 0$ for every representation Π and every unit vector $\xi \in \mathcal{H}_\pi$; so $\Pi(A) = 0$ for every Π.

By Lemma 4.1, $A = 0$. Hence for any nonzero $A \in \mathcal{B}$, there exists a pure state ρ of $\mathcal{B}$ with $\rho(A) \neq 0$. Thus, if Π_ρ is the cyclic representation of $\mathcal{B}$ induced by ρ, then $\Pi_\rho(A) \neq 0$ and Π_ρ is irreducible by Lemma 4.1. Therefore, $\mathcal{B}$ admits sufficiently many irreducible representations.

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