韓國數學教育學會誌<數學敎育>
987. 6. Vol. XXV, No. 3

Journal of the Korea Society of Mathematical Education
Jun. 1987. Vol. XXV, No. 3

On a Numerical Homotopy Method for Solving Systems of Nonlinear Equations

by Chin-Hong Park

— Dedicated to Professor Han Shick Park on his 60th birthday —

Abstract

.et $G: R^n \times R \longrightarrow R^n$ be defined by a Homotopy solving a system $F(x)=0$ of nonlinear equa-1s. For the vector v^k with $G'(u_k)v^k=0$, $\|v^k\|=1$ where u_k is one point in Zero Curve let $=v^k+\tau v^k$ be the first prediction for the next point u^{k+1}, $\tau \in (0,1)$. When $u_0{}^k$ is approaching too sely to some unwanted point, to follow the Zero Curve may occur the returning or cycling. One ition for it is discussed and the parametrizied Homotopy algorithm for solving $F(x)=0$ with it been established. Also some theorems by means of the regular value have been discussed for Zero Curves of $G(u)=0$ and some theorems for algorithm have been obtained.

Contents

0. Introduction

Let's denote

$$F(x)=0,\quad F: R^n \longrightarrow R^n$$

a system of nonlinear equations.

(a) **meaning or general idea;** This idea is based on the Homotopy concept in Topology. Tl names in Numerical analysis are known variously as "continuous Newton method", "continuati method", "Davidenko's method" or "Imbedding method". Here we use a Homotopy H.
Let f and $g: X \longrightarrow Y$ be continuous functions where X, Y are any spaces. We call H a *Homoto* from g to f if $H: X \times I \longrightarrow Y$ is a continuous map such that

$$H(x, 0)=g(x) \text{ and}$$
$$H(x, 1)=f(x) \text{ for every } x \in X.$$

We denote $H: g \simeq f$. where $I=[0, 1]$.
Let $\alpha_t=H(\cdot, t)$, i.e., $\alpha_t(x)=H(x, t)$, $x \in X, t \in I$, then the Homotopy H is seen to repres a family $\{\alpha_t : t \in I\}$ of maps from X to Y, varying continuously with t such that $\alpha_0=g$, $\alpha_1=$ This means that H gives a continuous deformation of g into f. Intuitively we can say that g be continuously transformed into f. For example, let $g(x)=x$ and $f(x)=0$ in R^n. Then g is motopic to f. i.e., define $H: R^n \times I \longrightarrow R^n$ by $H(x, t)=(1-t)x$, then it is clear. So our aim is imbed $F(x)$ into a Homotopy, where $F(x)$ is a nonlinear system of equations. For example we have $H(x, t)=F(x)+(t-1)F(x_0)$ where x_0 is a given point, when $t=1$ we have $H(x, 1)=F($ Hence $F(x)$ has been imbedded into a Homotopy and to find the solution of $F(x)$, all we h to do are to read the value of x when $t=1$. Here the only interesting curve is $H(x, t)=0$, called Zero Curve.

(b) **historical background;** The first application of this tool to Numerical solution of nonli equations is attribute to E. Lahaye (1934, 1935) and D.F. Davidenko (1953). Lahaye's appr is a locally convergent, iterative continuation method while that of Davidenko is continuation m od by differentiation. The more references before 1970 can be found in (01) but most of t are not discussed about Algorithm in detail to implement the continuation method. After 1970, notable facts that can be discussed can be found in (R1), (R2), (K1), (CMY1), (W1), (GG1 (S3) and (L1), (M1), (S1, 2), etc,... Moreover Leder, R. Menzel and H. Schwetlick l discussed the Algorithm and theoretical aspect in detail about both Singular and Nonsingular c

(c) **the reason why we use it;** Many iterative techniques and Newton method for the sol of nonlinear equations have the drawback that the convergence depends on only good approx tion. If no approximation roots are known, most of the classical methods may be of little use. Newton method could cycle or could blow up when the Jacobian Matrix at some point is sing The Imbedding method has the advantages of producing solutions over a large range of the pendent variables and is used as a tool in overcoming the local convergence of iterative proc Also the Imbedding methods may be considered as a possibility to widen the Domain of co gence or, from another point of view, as a procedure to obtain sufficiently close starting poin

(d) **how to find the solutions of $F(x)=0$ by using Homotopy;** Choose a Homotopy H s at $t=0$, the solution of $H(x,t)=0$ is a known point $x_0 \in R^n$, while at $t=1$, the solution x^* of $H(x,t)=0$ solves $F(x)=0$. For example,

$$H(x,t)=tF(x)+(1-t)(x-x_0)$$

satisfies the conditions. So we start at $x=x_0$ and follow the curve of $H(x,t)=0$, Zero Curve until $t=1$. At last when $t=1$, we read the value of x. The most common Homotopies used are

$$H(x,t)=tF(x)+(1-t)(x-x_0)$$
$$H(x,t)=F(x)+(t-1)F(x_0).$$

The essence of the imbedding method is "path-following". Theoretically this is very simple but the problem is how a method is implemented by computer program efficiently, i.e., how do we follow the zero curve? Because the end point ($t=1$) of the zero curve is of more interest while the curve itself is of lesser interest, we think that Lahaye's approach is more appropriate. In general there are many solutions of $F(x)=0$. The details for this can be found in H.B. Keller or (GG1). We note that following the zero curve in case of the turning points, we have to follow it very closely.

(e) **existence of solution curve with $H(x,t)=0$, $t \in [0,1]$;** The existence was discussed in (O1), (K1), (CYM1), and (S1). In (O1) the existence is described as follows: Let $F: R^n \longrightarrow R^n$ be a C^1-map on R^n and assume that $F'(x)$ is nonsingular for all $x \in R^n$. Assume that $\| F'(x)^{-1} \| \leq M$ for some $M>0$ for all $x \in R^n$. Then for any fixed $x_0 \in R^n$ there exists a unique C^1-map $z: [0,1] \longrightarrow R^n$ such that $H(z(t),t)=0$ for all $t \in [0,1]$. Moreover $z(t)=-F'(z(t))^{-1}F(x_0)$ for all $t \in [0,1]$, $x(0)=x_0$, where $H: D \times [0,1] \longrightarrow R^n$ is a Homotopy given by

$$H(x,t)=F(x)+(t-1)F(x_0), D \subset R^n.$$

The conditions of F for existence have been replaced by the conditions of H in (S1), i.e., if H satisfies the above conditions, there exists a unique C^1-map z: $[0,1] \longrightarrow R^n$ such that $H(z(t),t)=0$ for all $t \in [0,1]$. In (S1) it's called "a regular imbedding for F to $x_0 \in R^n$". The most impor tant thing in a regular imbedding is that $\partial_1 H(x,t)$ is nonsingular where $\partial_1 H(x,t)=\partial H(x,t)/\partial x$, $H(x,t)=F(x)+(t-1)F(x_0)$. If $\partial_1 H(x,t)$ is singular at some points, it is called "a singular imbedding for F to $x_0 \in R^n$ with some conditions" in (M1). In that case the requirement to regularity of $\partial_1 H$ is replaced by the condition of linearly independent decompositions from $G'(u)=(\partial_1 H, \partial_2 H)$. The above conditions of existence in (O1) are strong. The another representation of existence can be expressed by means of regular value. Let $H: R^{n+1} \longrightarrow R^n$ be a C^1-map and let 0 be a regular value of H. Then $H^{-1}(0)$ is a C^1-submanifold with dimension 1, i.e., this means the existence of zero curves. In (CMY1) this was expressed by the general, parametrized Sard's theorem. For the more details, see (CMY1).

(f) **problems of imbedding method;** One of the crucial problems encountered when we use an imbedding method (continuation method) is the selection of the stepsize. A step that is too large may result in the initial estimate being outside the convergent region of the iterative process and result in a failure of the process, or may pass over the critical points in the solution. In (M1) and (R1), (S1) the automatic, variable stepsizes have been used. In (R1) the following problems were pointed out: efficient design of steplength selection, analysis and control of the accuracy and stability of the computational solution, control of the computational cost (unexplored part).

1. Assumptions and Notations.

Let's follow R. Menzel and H. Schwetlick's assumptions with the slight modifications.

(1) there exists a C^2-map $W:(-\varepsilon, 1+\varepsilon)\longrightarrow R^n\times R$ such that the following conditions are satisfied:

(a) $W(0)=(x_0, 0)^T$, $W(1)=(x^*, 1)^T$

$\dot{t}(0)>0$, $\quad t(s)<1$ for all $s\in[0, 1)$

(b) $\dot{W}(s)=\dfrac{dW}{ds}\neq 0$, $GW(s)=0$ for all $s\in[0, 1]$

where $\varepsilon>0$

(2) (a) G is defined by a Homotopy H, i.e., $G(x, t)=H(x, t)$ for all $(x, t)\in R^n\times R$.

(b) $G'(u)$ satisfies the Lipschitz condition on a neighborhood U of Z_0 where $Z_0=W([0, 1])$, i.e., for some $L>0$

$\|G'(u_1)-G'(u_2)\|\leq L\|u_1-u_2\|$ for all $u_1, u_2\in U$.

(c) $G'(u)$ has full-rank on Z_0, i.e., rank $G'(u)=n$ for all $u\in Z_0$.

Note: For (b) of (2) we can assume that $G: R^n\times R\longrightarrow R^n$ is a C^2-map in a neighborhood U of Z_0.

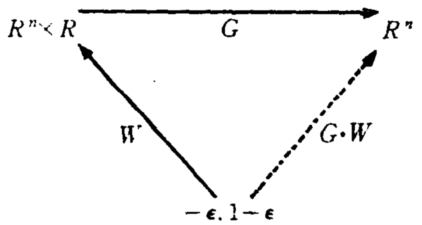
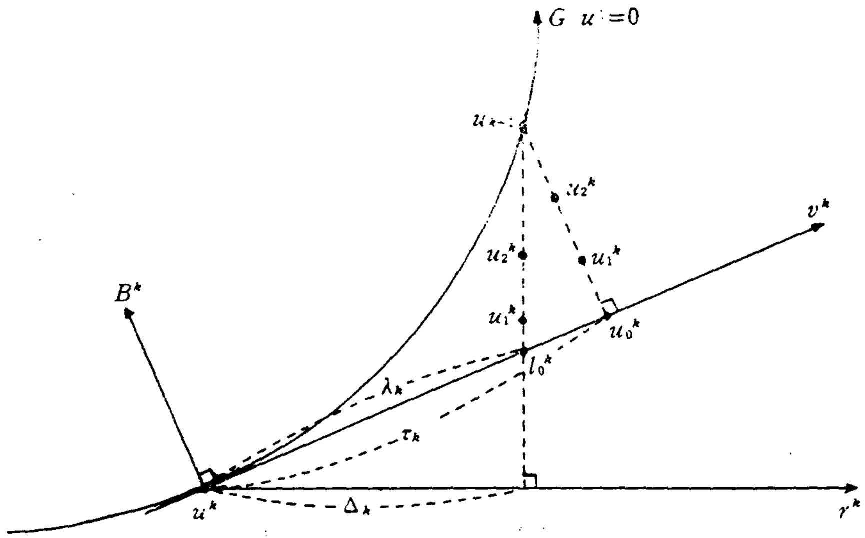


Fig. 1



Notations and Meaning:

u_k : true value of $G(u)=0$

u^k : approximation of u_k and $u^k=(x^k, t_k)$.
$e_n=(0, 0, \cdots, 0, 1)$: unit vector with n-th coordinate 1.
$(e_{n+1})^T u^k = t_k$
v^k : tangent vector with $G'(u^k)v^k=0$, $\|v^k\|=1$ at u^k.
r^k : unit vector with $(r^k)^T(u_{i+1}{}^k - u_i{}^k)=0$, $i=0,1,2,\cdots$
A^+ : pseudoinverse of a matrix A, i.e., $A^T(AA^T)^{-1}$
A^T : transpose of a matrix A
$L(R^m, R^n)=\{f|f : R^m \longrightarrow R^n$ is a linear map$\}$
$\quad$ =the linear space of real $n\times m$ matrices.
$u_0{}^k = u^k + \tau_k v^k$, the first predictor with $(v^k)^T(u-u^k)=\tau_k$ *on* v^k
$l_0{}^k = u^k + \lambda_k v^k$, the first predictor with $(r^k)^T(u_{i+1}{}^k - u_i{}^k)=0$
B^k : a vector orthogonal to v^k with $B^k\cdot v^{k-1}<0$
$\Pi_* = \{u\in R^{n+1} : (e_{n+1})^T u = 1\}$, hyperplane.
$\Pi_k = \{u\in R^{n+1} : (v^k)^T(u-u^k)=\tau_k\}$, normal plane with respect to v^k with distance τ_k from u^k.
$t_{k+1} = (e_{n+1})^T u_0{}^k$

2. Theoretical Background for Assumption.

Definition 2.1. (a) Let $f : R^m \longrightarrow R^n$ be a C^1-map. We call $y\in R^n$ a *regular value* if range $Df(x)=R^n$ (or rank $f'(x)=n$) for all $x\in f^{-1}(y)$. Otherwise y is called a *critical valve*. (b) we call M a *n-dimensional manifold* if each point of M has an open ***neighborhood*** homeomorphic to R^n.

Lemma 2.2. (1) *Let M and N be manifolds of dimensions m, n, respectively. Let $f : M\longrightarrow N$ be a C^r-map, $r\geq 1$. If $y\in f(M)$ is a regular value then $f^{-1}(y)$ is a C^r-submanifold of M with dim$(m-n)$.*

(2) (*Morse-Sard*) *Let M and N be manifolds of dimensions m, n and $f : M\longrightarrow N$ be a C^r-map. Let C be the set of critical points of f. If $r>max\{0, m-n\}$, then $f(C)$ has measure zero in N (i.e., almost all $y\in N$ is regular value).*

(3) (*Chow-Mallet Paret-Yorke*) *Let $U\subset R^m$ and $V\subset R^p$ be open sets and $\varphi : U\times V\longrightarrow R^n$ be C^r-map, $r>max\{0, m-n\}$. If $0\in R^n$ is a regular value of φ then for almost all $a\in V$, 0 is a regular value of φ_a where $\varphi_a(x)=\varphi(x, a)$.*

Note: (1) See pp. 22 in (H1). When M is compact manifold $f^{-1}(y)$ is finite set (possibly empty) see (M3). (2) See pp. 69 in (H1). (3) This is the more general, parametrized Sard's theorem. The details are in (CMY1).

The following theorem can be proved easily by virtue of (3) of Lemma 2.2.

Theorem 2.3. (parametrized Sard's theorem on Manifolds) *Let M, S and N be manifolds with dimensions m, p and n respectively. Let $U\subset M$ and $V\subset S$ be open sets.*

Let $\varphi : U\times V\longrightarrow N$ be a C^r-map, $r>max\{0, m-n\}$. If $0\in N$ is a regular value of φ then for almost every $a\in V$, 0 is a regular value of φ_a where $\varphi_a(x)=\varphi(x, a)$.

Proof. Note that every open set of n dimensional manifold is a n-dimensional manifold. For almost every $a\in V$, it is enough to show that rank $\varphi_a'(x_0)=n$ for every $x_0\in\varphi_a{}^{-1}(0)$. Suppose that

$(x_0, a) \in U \times V$. Then there exist open sets U_1, V_1 of x_0, a respectively, such that U_1 and V_1 are homeomorphic to open sets K, L, respectively, where $K \subset R^m$, $L \subset R^p$. Also since $0 \in N$ there exists an open subset $W \subset N$ of 0 such that W is homeomorphic to R^n. Let $U_2 \times V_2 = \varphi^{-1}(W) \cap (U_1 \times V_1)$.

Then $(x_0, a) \in U_2 \times V_2$ because of $x_0 \in \varphi_a^{-1}(0)$. Let $\alpha = \varphi | U_2 \times V_2$. Then $\alpha : U_2 \times V_2 \longrightarrow W$ is a C^r-map and 0 is a regular value of α. By (3) of Lemma 2.2, 0 is regular value of α_a. Since $x_0 \in \alpha_a^{-1}(0)$ rank $\alpha_a'(x_0) = n$. Now $\alpha_a(x) = \varphi_a(x)$ for $x \in U_2$, and $\alpha_a'(x) = \varphi_a'(x)$. Also $\alpha_a'(x_0) = \varphi_a'(x_0)$. Hence rank $\varphi_a'(x_0) = $ rank $\alpha_a'(x_0) = n$.

For simplicity let $B^n = \{x \in R^n : |x| < 1\}$ and we will consider the regularizing Homotopy

$$H(x, t) = tF(x) + (1-t)(x - x_0).$$

Theorem 2.4. *Let* $F : \bar{B}^n \longrightarrow \bar{B}^n$ *be a* C^2*-map and* $\varphi : B^n \times (0,1) \times B^n \longrightarrow R^n$ *be a* C^2*-map given by*

$$\varphi(x, t, a) = tF(x) + (1-t)(x - a).$$

Let Z_a *be the component of* $\varphi_a^{-1}(0) \cap B^n \times (0,1)$ *whose closure contains* $(a, 0)$ *where* $\varphi_a(x, t) = \varphi(x, t, a)$. *Then the following facts are held:*

(1) $0 \in R^n$ *is a regular value of* φ. *(i.e., the Jacobian matrix of* $\varphi(x, t, a)$ *has full rank on* $\varphi^{-1}(0)$*).*

(2) $0 \in R^n$ *is a regular value of* φ_a *for almost every* $a \in B^n$.

(3) $\varphi_a^{-1}(0)$ *is a* C^2*-submanifold of* $B^n \times (0,1)$ *with dimension* 1 *for almost all* $a \in B^n$.

(4) *For almost every* $a \in B^n$, Z_a *is a smooth curve in* $B^n \times (0,1)$ *connecting between* $(a, 0)$ *and a zero point of* $F(x) = 0$ *at* $t = 1$. *This means that* $Z_a \subset \bar{B}^n \times (0,1)$ *is not diffeomorphic to a circle and has no limit points on* $\partial B^n \times (0,1)$.

(5) *If* $F'(x)$ *is nonsingular for every zero point* x *of* $F(x)$, *then* $\bar{Z}_a$ *is a smooth curve in* $\bar{B}^n \times [0,1]$ *and has finite arc length.*

(6) *For summary of* $\varphi_a^{-1}(0)$, $\varphi_a^{-1}(0)$ *consists of* (a) *a finite number of closed loops in* $\bar{B}^n \times (0,1)$, (b) *a finite number of arcs in* $\bar{B}^n \times (0,1)$ *with end points in* $\bar{B}^n \times \{1\}$, (c) Z_a.

Remark. (2.1) If condition of (5) is satisfied, then the curves of (a), (b), and (c) of (6) have finite lengths. Moreover, these curves are disjoint.

The proof of this theorem can be found in (CMY1) and (W1), but in (CMY1) and (W1) the fixed points were discussed. In the same way this theorem can be proved for $F(x) = 0$.

(2.2) The more references for this theorem can be found in (K1), (GG1) and (GG2). The main theorem of (K1) and (GG1) was almost same under Smale's boundary condition. In (GG1) and (GG2) the theorems for algorithm were discussed and in (K1) the main idea of algorithm is the same as that of (M1).

(2.3) In (CMY1) they say that if we choose a point at random from V (or B^n), the probability is one that each component of $\varphi_a^{-1}(0)$ is a smooth curve, i.e., the existence of such curves is guaranteed with probability one.

(2.4) If we let $G = \varphi_{x_0}$ for $x_0 \in B^n$ then we will get $Z_0 = Z_{x_0}$ and the justification of assumptions turns out to be clear where $x_0 \in B^n - A$, A is the set of measure zero with respect to n-dimensional Lebesgue measure.

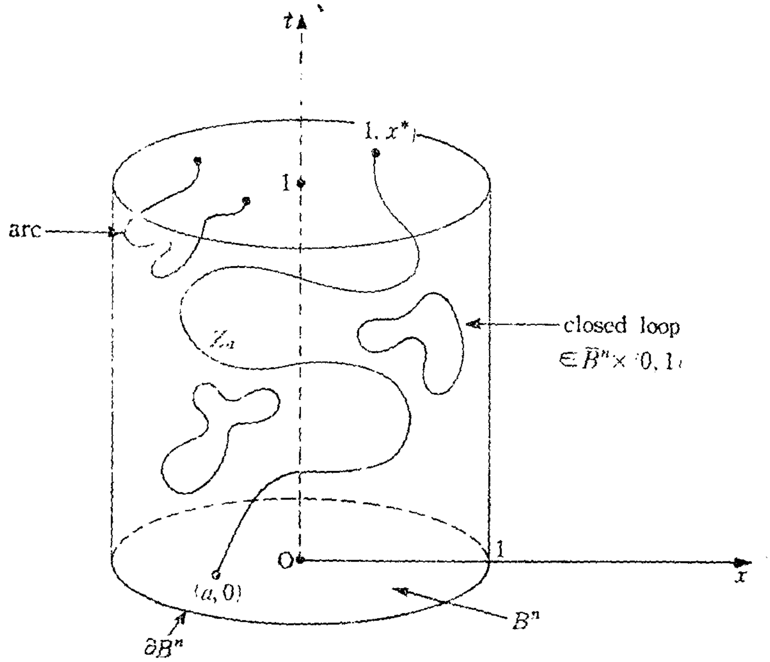
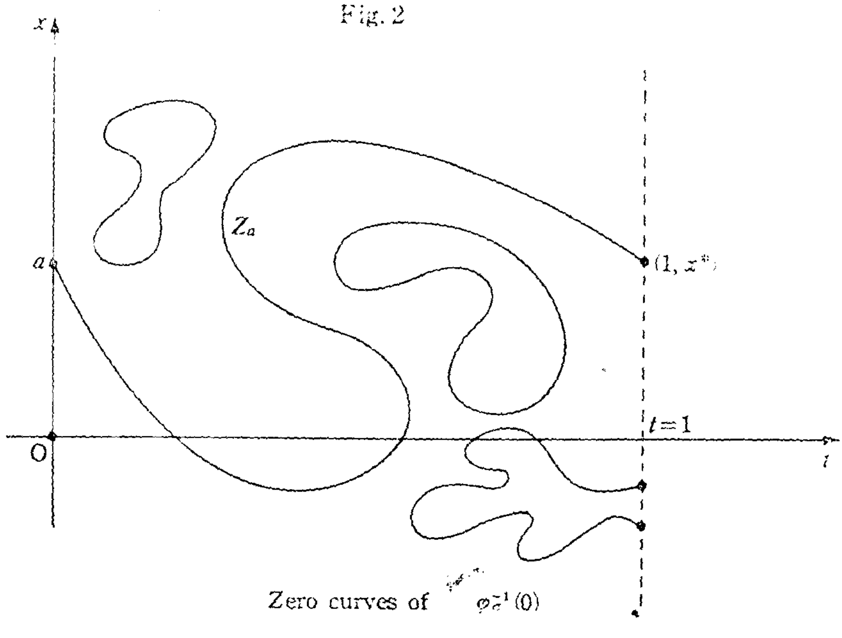


Fig. 2



Zero curves of $\varphi_a^{-1}(0)$

3. The Results obtained from Assumptions.

We can see that the following facts hold under assumptions.

Proposition 3.1. (1) *There exists* $\delta_0>0$ *such that* $Z(\delta_0)=\{u\in U: \|u-u_0\|<\delta_0,\ u_0\in Z_0\}\subset U$ *is compact neighborhood of* Z_0 *with* $Z(\delta_0)\subset\overline{Z}(\delta_0)\subset U$ *where* U *is an open neighborhood of* Z_0, $Z_0=W([0,1])$.

(2) $\|G'(u)^+\|\leq M$ *for some* $M>0$ *for every* $u\in Z_0$, *where* $G'(u)^+=G'(u)^T(G'(u)G'(u)^T)^{-1}$ *is pseudoinverse to* $G'(u)$.

For (1) since R^n is locally compact the result follows immediately.

The following Lemma is the result obtained from the Linear algebra and also it is in (MS1).

Lemma 3.2. *Given a matrix* $A\in L(R^{n+1},R^n)$ *with rank* $(A)=n$,

(1) $Av=0$ *has a solution* $v\in R^{n+1}$ *with* $\|v\|=1$ *and the form* λv *is another solution of* $Av=0$, $\lambda\in R$.

(2) *let* $B=\begin{pmatrix}A\\ w^T\end{pmatrix}\in L(R^{n+1})$, $w\in R^{n+1}$ *with* $\|w\|=1$.

If $w^Tv\neq 0$ *with* $Av=0$, $\|v\|=1$ *then*

(*a*) B *is regular*

(*b*) $B^{-1}=\left[\left(I-\frac{vw^T}{v^Tw}\right)A^+ \;\vdots\; \frac{v}{v^Tw}\right]$

and $\|B^{-1}\|\leq\frac{2}{|w^Tv|}max\{\|A^+\|,1\}$,

where $A^+=A^T(AA^T)^{-1}$ *is pseudoinverse to* A.

The following Lemma is rewritten from (MS1).

Lemma 3.3. *Suppose that the assumptions are satisfied. Let* $Z_0=w([0,1])$, $u_0\in Z_0$ *an* $G'(u_0)v_0=0$ *with* $\|v_0\|=1$.

Define $d:\overline{Z}(\delta_0)\longrightarrow L(R^{n+1})$ *by* $\quad d(u)=\begin{bmatrix}G'(u)\\ v_0{}^T\end{bmatrix}$

Then there exist $\delta_1>0$, $\delta_1\leq\delta_0$, $\eta>0$, $c\in(0,1)$ *such that for every* $u\in\bar{S}(u_0;\delta_1)=\{u: \|u-u_0\|\leq\delta_1\}$

(1) $d(u)=\begin{bmatrix}G'(u)\\ v_0{}^T\end{bmatrix}$ *is regular*

(2) *rank* $G'(u)=n$ *(full rank) and there exists exactly one* $v\in R^{n+1}$ *with* $G'(u)v=0$, $\|v\|=$ *and for* v, $(v_0)^Tv\geq c>0$

(3) $\|v-v_0\|\leq\eta\|u-u_0\|$.

Proposition 3.4. *For the compact neighborhood* $Z(\delta_0)$ *of* Z_0, *there exists a full ra neighborhood* $Z(\delta_1)\subset\overline{Z}(\delta_0)$ *of* Z_0 *in the sense that rank* $G'(u)=n$ *for every* $u\in Z(\delta_1)$.

This is obvious from the above Lemma, letting

$$Z(\delta_1)=\bigcup_{u_0\in z_0}\bar{S}(u_0;\delta_1).$$

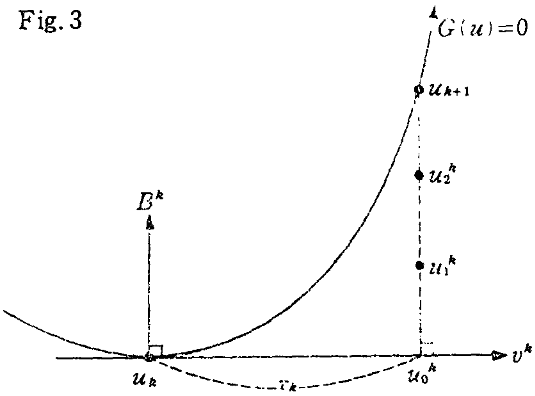
For (2) since $G'(u)$ has full rank on $Z(\delta_1)$ the tangent direction $v\in R^{n+1}$ is apart from si uniquely determined by $G'(u)v=0$, $\|v\|=1$.

Let $u_k \in Z_0$ and $\tau_k \in (0, 1]$. Then we can consider one of the following two systems of equations to find the approximation u^{k+1} of u_{k+1}, the next point of u_k with Newton method.

$$\begin{cases} G(u) = G(u_k) \\ (v^k)^T(u-u_k) = \tau_k \end{cases}$$

$$\begin{cases} G(u) = G(u_k) \\ (v^k)^T(u-u_i^k) = 0 \quad i=0, 1, 2, \cdots \end{cases}$$

Fig. 3



Proposition 3. 5.

Let $D(u) = G(u) - G(u_k)$ *and*

$E(u) = (v^k)^T(u-u_k) - \tau_k$ *for every* $u \in \bar{S}(u_k;\ \delta_1)$.

Define $d_k : R^{n+1} \longrightarrow R^n$ *by* $d_k(u) = \begin{pmatrix} D(u) \\ E(u) \end{pmatrix}$. *Then*

(1) $d_k'(u) = \begin{bmatrix} G'(u) \\ (v_k)^T \end{bmatrix}$ *is regular on* $\bar{S}(u_k;\ \delta_1)$

(2) $d_k'^{-1}(u) = \left[\left(I - \frac{v(v^k)^T}{v^T v^k}\right) G'(u)^+ \ \vdots\ \frac{v}{v^T v^k}\right]$,

where $G'(u)^+$ *is pseudoinverse to* $G'(u)$, $G'(u)v=0$ *with* $\|v\|=1$.

Proof. (1) Note that rank $G'(u)=n$ for all $u \in \bar{S}(u_k;\ \delta_1)$. By Lemma 3. 3, there is exactly one v with $G'(u)v=0$, $\|v\|=1$ *on* $\bar{S}(u_k;\ \delta_1)$ and for v, $(v^k)^T v > 0$. It is held by Lemma 3. 2. (2). See Lemma 3. 2.

Remark.

(a) If we want to find the approximation u^{k+1} with Newton method, we can use the result of Proposition 3. 5.

Note that $d_k(u)=0 \iff \begin{cases} G(u) = G(u_k) \\ (v^k)^T(u-u_k) = \tau_k. \end{cases}$

So we can get the next:

$u_{i+1}^k = u_i^k - d_k'^{-1}(u_i^k) d_k(u_i^k)$, $i=0, 1, \cdots$

$d_k'(u_i^k)^{-1} d_k(u_i^k) = \left(I - \frac{v_i^k (v^k)^T}{(v_i^k)^T v^k}\right) G'(u_i^k)^+ \left[G(u_i^k) - G(u_k)\right]$,

where $G'(u_i^k) v_i^k = 0$ with $\|v_i^k\| = 1$.

(b) Let $(r^k)^T v > 0$ with $G'(u)v=0$, $\|v\|=1$ and

$D(u) = G(u) - G(u_k)$

$E(u) = (r^k)^T(u-u_k) - \triangle_k$ for all $u \in \bar{S}(u_k;\ \delta_1)$.

Then we can get the same result about $d_k(u) = \begin{bmatrix} D(u) \\ E(u) \end{bmatrix}$.

(c) For simplicity of (a),

$$d_k'(u_i^k)^{-1}d_k(u_i^k)=(I-v^k(v^k)^T)G'(u^k)^+\ [G(u_i^k)-G(u^k)],$$

where $G'(u)^+$ is a pseudoinverse to $G'(u)$ and v is a vector with $G'(u)v=0$, $\|v\|=1$.

Let's define the following things in similar way like those in (S1) and (M1) by means of parameters in Homotopy.

Definition 3.6.

(a) $H_{x_0}: R^n\times J\longrightarrow R^n$ is *one parameter imbedding map* for F with respect to $x_0\in R^n$ if the following conditions are satisfied:

(1) $H_{x_0}(x_0,0)=0$

(2) $H_{x_0}(x,1)=F(x)$ for all $x\in R^n$.

where $F: R^n\longrightarrow R^n$ is a map and $J=(-\varepsilon,1+\varepsilon)$, $\varepsilon>0$.

Notation: If x_0 is fixed, let's denote $H_{x_0}=H$.

(b) $\varphi: R^n\times J\times R^n\longrightarrow R^n$ is a *family of imbeddings* for F if φ_a is an imbedding for F to a, for every $a\in R^n$, i.e., $\varphi_a(a,0)=0$, $\varphi_a(x,1)=F(x)$ for all $x\in R^n$ and note that $\varphi_a(x,t)=\varphi(x,t,a)$.

(c) $G_{x_0}: R^n\times R\longrightarrow R^n$ is a *weakly singular*, *C^2-imbedding* for F to $x_0\in R^n$ if the following conditions are satisfied:

(1) G_{x_0} is defined by a Homotopy H_{x_0}. i.e.,

$$G_{x_0}(x,t)=H_{x_0}(x,t)\ \text{for every}\ (x,t)\in R^n\times R$$

(2) there exists a *C^2-map* $W:[0,1]\longrightarrow R^{n+1}$ such that $W(0)=(x_0,0)^T$, $W(1)=(x^*,1)^T$, $t(s)<1$ for all $s\in[0,1)$ and $GW(s)=0$, $W(s)=\frac{dW}{ds}\neq 0$ for every $s\in[0,1]$. Note that the parameter s is different from the Homotopy parameter t in general.

(3) G_{x_0} is a C^2-map on a *neighborhood* U of W.

(4) G_{x_0}, has a full rank on W, i.e., rank $G'_{x_0}(W(s))=n$ for $s\in[0,1]$. Note that $G_{x_0}(x_0,0)=0$, $G_{x_0}(x,1)=H_{x_0}(x,1)=F(x)$.

Notation: If x_0 is fixed, let $G_{x_0}=G$.

(d) $\varphi: R^n\times J\times R^n\longrightarrow R^n$ is a *weakly singular*, *C^2-family of imbeddings* for F if $\varphi_a: R^n\times J\longrightarrow R^n$ is a weakly singular, C^2-imbedding for F with respect to a for all $a\in R^n$, i.e., φ_a satisfies the conditions (1), (2), (3) and (4) of (c).

(e) $\alpha_k: (-\bar{\tau},\bar{\tau})\longrightarrow R^{n+1}$ is a *local parametrization of solutions* of $\begin{cases}G(u)=G(u_k)\\(v^k)^T(u-u_k)=\tau\end{cases}$ ......(*)

if for each $\tau\in(-\bar{\tau},\bar{\tau})$ there exists a unique solution

$\alpha_k(\tau)$ of (*) and α_k is a C^1-map.

The following fact is rewritten by using the local parametrization from (M1). The proof i easy by using Lemma 3.5 of (M1) and implicit function theorem.

Proposition 3.6. *Let G: $R^n\times R\longrightarrow R^n$ be a weakly singular, C^2-imbedding for F with respec to $x_0\in R^n$. Then for each $u^k\in Z(\delta_1)$ there exist $\tau>0$, $q>0$ such that $\alpha_k: (-\bar{\tau},\bar{\tau})\longrightarrow R^n$ is local parametrization of solutions of (*) and*

$$\|\alpha_k(\tau)-(u^k+\tau v^k)\|\leq q\tau^2\ \textit{for all}\ \tau\in[-\bar{\tau},\bar{\tau}].$$

4. Implementable Algorithm.

The algorithm for finding a root of $F(x)=0$ is as follows: start with $t=0$, $x_0 \in R^n$ and follow the zero curve Z_0 of $G_{x_0}(x,t)=G(x,t)$ emanating from $(x_0, 0)$. By the previous theorems Z_0 reaches a zero x^* of F. The algorithm discussed here is based on the algorithm of *H.* Schwetlick and R. Menzel. But there will be much modifications and developments. In imbedding methods, the algorithms can be carried out without using the global constants such as Lipschitzian and regularity constants. The implementation is not as easy as it appears. Here mainly Lahaye's approach will be discussed for algorithm because Davidenko's approach requires the closer approximation of zero curve and hence the more computing time but in case of the turning points we have to follow the zero curve very closely. Also the failure of Menzel algorithm will be discussed. When the first predictor u_0^k is approached too closely to some unwanted point of zero curve, the failure may occure. In fact we think that our aim is x^* rather than the curve itself. The concept of parametrized Homotopy will be used for algorithm and the basic Tangent algorithm will be established. The method using the tangent vector, tangent plane and normal plane was also discussed in H.B. Keller. There to estimate the stepsize, he used Newton-Kantorovich theorem.

By the previous theorem, the unique tangent vector v^k is determined for $u^k \in R^n \times R$ from $G'(u^k)v^k=0$ with $\|v^k\|=1$ and $u_0^k=u^k+\tau_k v^k$ is chosen as the approximation for the next point u^{k+1}. If u_0^k does not satisfy the inquality $\|G(u_0^k)\| \leq \|G(u^k)\|+\tau_k\varepsilon$, τ_k will be reduced by $\alpha \in (0,1)$, i.e., $\tau_k := \alpha\tau_k$. If u_0^k satisfies the inquality u_0^k is accepted as a successor of $u^k=(x^k, t_k)$ where $\varepsilon>0$ is a given number and $\tau_k \in [\tau_{k-1}, \bar{\tau}]$, $0<\tau_{-1} \leq \bar{\tau} \leq 1$.

4.1. Some Theorem for Convergence.

Lemma 4.1.1. *Let $E \subset R^n$ be an open set. Let $F: E \subset R^n \longrightarrow R^n$ be a C^2-map with $F(a)=0$, $a \in E$ and $C(a)$ be a compact set containing a with $C(a) \subset E$. Also assume that $\|F'(x)^{-1}\| \leq M$ for some $M>0$ on $C(a)$. Then for any $\mu \in (0,1)$ there exists $r=r(\mu)>0$ such that*

(*a*) *Newton sequence $x_{n+1} := x_n - F'(x_n)^{-1}F(x_n)$ is well defined for any x_0 with $\|x_0-a\| \leq r$, $n=0,1,\cdots$*

(*b*) $\|F(x_{n+1})\| \leq \mu \|F(x_n)\|$ *for all n*

(*c*) *$\{x_n\}$ converges to a quadratically.*

Proof. Let $\bar{S}_0=\{x \in C(a) : \|x-a\| \leq r_0\} \subset C(a)$ be the closed ball for some $r_0>0$. Let $\|F'(x)\| \leq M_1$ and $\|F''(x)\| \leq M_2$ for some $M_1>0$, $M_2>0$ on S_0.
Let

$$r=\min\left\{\frac{r_0}{2}, \frac{2}{MM_2}, \frac{2\mu}{M_1M_2M^2}\right\}$$

$$\bar{S}=\{x \in R^n : \|x-a\| \leq r\}$$

$$y=x-F'(x)^{-1}F(x) \text{ for } x \in \bar{S}.$$

Then y is well defined, $\|y-a\| \leq \frac{2}{MM_2}\|x-a\|^2 \leq r$, i.e., $y \in \bar{S}$ if $x \in \bar{S}$ and $\|F(y)\| \leq \mu \|F(x)\|$. For the more details of proof see (S1).

Note. PROCEDURE CONV-TEST-TO-X* is based on Theorem 4.1.2. If u^{k+1} with $t_{k+1}=1$ or one point of Newton iterates is contained in $\|x-x^*\| \leq r$, the convergence of $x_n \longrightarrow x^*$ is guaranteed.

4.2. Analysis and Meaning of Algorithm.

Lemma 4.2.1. *Assume that A and B are nonzero vectors. Then there exists a unique number r such that $A-rB$ is orthogonal to B, where* $r=\frac{A\cdot B}{\|B\|^2}$, *and "." means inner product.*

Calculation of the vector B^k orthogonal to v^k :

$k=0$: See Fig.4. Compute v^0 with $G'(u^0)v^0=0$, $\|v^0\|=1$. If $(v^0)^T e_{n+1}<0$ then $v^0:=-v^0$.

Let $B^0:=e_{n+1}-rv^0$, $r=(e_{n+1})^T v^0$ according to Lemma 4.2.1. If $(e_{n+1})^T B^0<0$ then $B^0:=-B^0$.

$k\neq 0$: Assume that u^{k-1}, B^{k-1} and v^{k-1} were obtained. Also we assume that

$(B^{k-1})^T(u_1^{k-1}-u^{k-1})>0$ and

$$\|G(u_0^{k-1})\|\leq\|G(u^{k-1})\|+\varepsilon\tau_{k-1}.$$

Note that u_1^{k-1} can be obtained from Remark of Proposition 3.5. Compute v^k with $G'(u^k)v^k=0$, $\|v^k\|=1$. By Lemma 4.2.1, let $B^k:=(u_1^{k-1}-u^k)-rv^k$, $r=(u_1^{k-1}-u^k)v^k$.

Using the Inverse function theorem and the fact that R^n is locally compact, we can get the following theorem.

Theorem 4.1.2. *Let $E\subset R^n$ be an open set. Let $F: E\subset R^n\longrightarrow R^n$ be a C^2-map with $F(x^*)=0$, $x^*\in E$. Assume that $F'(x^*)$ is nonsingular. Then for any $\mu\in(0,1)$ there exists $r=r(\mu)>0$ such that*

(*a*) *Newton sequence $x_{n+1}:=x_n-F'(x_n)^{-1}F(x_n)$ is well defined for any x_0 with $\|x_0-x^*\|\leq r$*

(*b*) $\|F(x_{n+1})\|\leq\mu\|F(x_n)\|$ *for all n.*

(*c*) $\{x_n\}$ *converges to x^* quadratically.*

Proof. Since $F'(x^*)$ is nonsingular, according to the Inverse function theorem there exist open sets $U\subset E$, $V\subset R^n$ of x^* and $F(x^*)$ respectively, such that $F|U: U\cong V$ (homeo.) and $F|U$ has a C_1-inverse function $g: V\longrightarrow U$. Since R^n is locally compact, there is a compact set $C(x^*)\subset U$ of x^*. Also $F(C(x^*))\subset V$ is a compact set. Now

$\|F'(x)^{-1}\|=\|g'(F(x))\|\leq M$ for some $M>0$, for every $x\in C(x^*)$. By Lemma 4.1.1. the desired results are held.

$$\text{If } (B^k)^T v^{k-1}>0 \text{ then } B^k:=-B^k.$$

CONV-TEST-TO-X* :

According to Theorem 4.1.2, since $F'(x^*)$ is nonsingular there exists a closed *neighborhood* $\bar{S}_r$ of x^* such that if a certain point x_0 is contained in $\bar{S}_r$, the convergence of Newton sequence and $\|F(x_{n+1})\|\leq\mu\|F(x_n)\|$ are guaranteed for all n and for every $\mu\in(0,1)$. Hence when $t=1$, we test this possibility. The procedure can be described as follows: See Fig.5., where $\bar{S}_r=\{x\in \mathrm{R}^n : \|x-x^*\|\leq r\}$.

PRED. $\quad w:=\sum_{i=1}^{n}(e_i)^T u_0^k$; $j:=0$; $x_0:=w$

CORRECT. $\quad x_{j+1}:=x_j-F'(x_j)^{-1}F(x_j)$

(or Runge-Kutta method of order 4)

TEST $\quad$ IF $\|F(x_{j+1})\|\leq\mu\|F(x_j)\|$

THEN IF $\|F(x_{j+1})\|\leq\xi$

Fig. 4

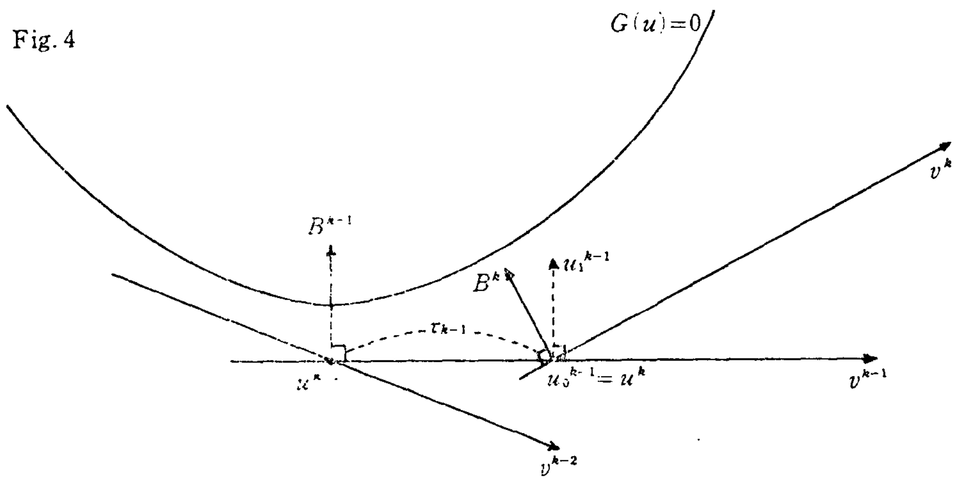
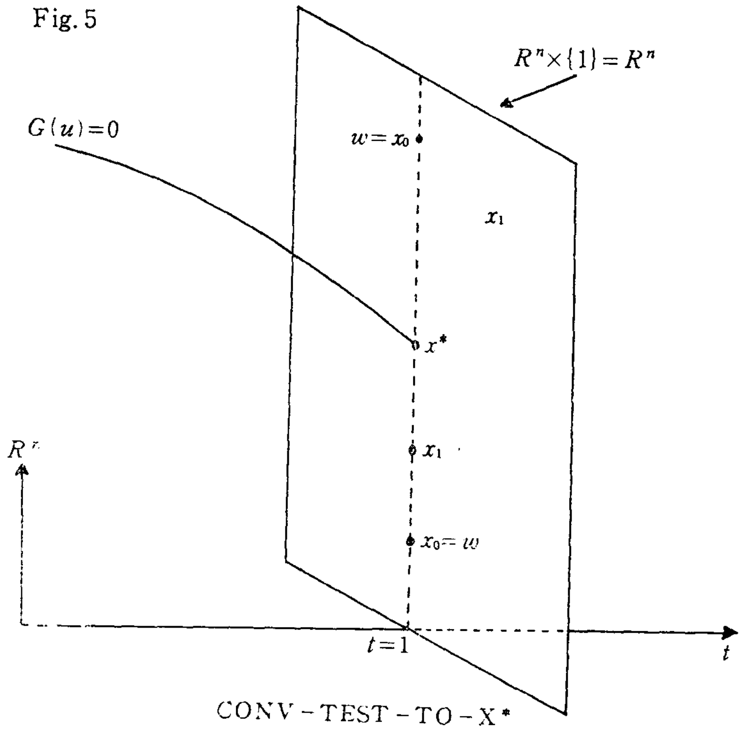
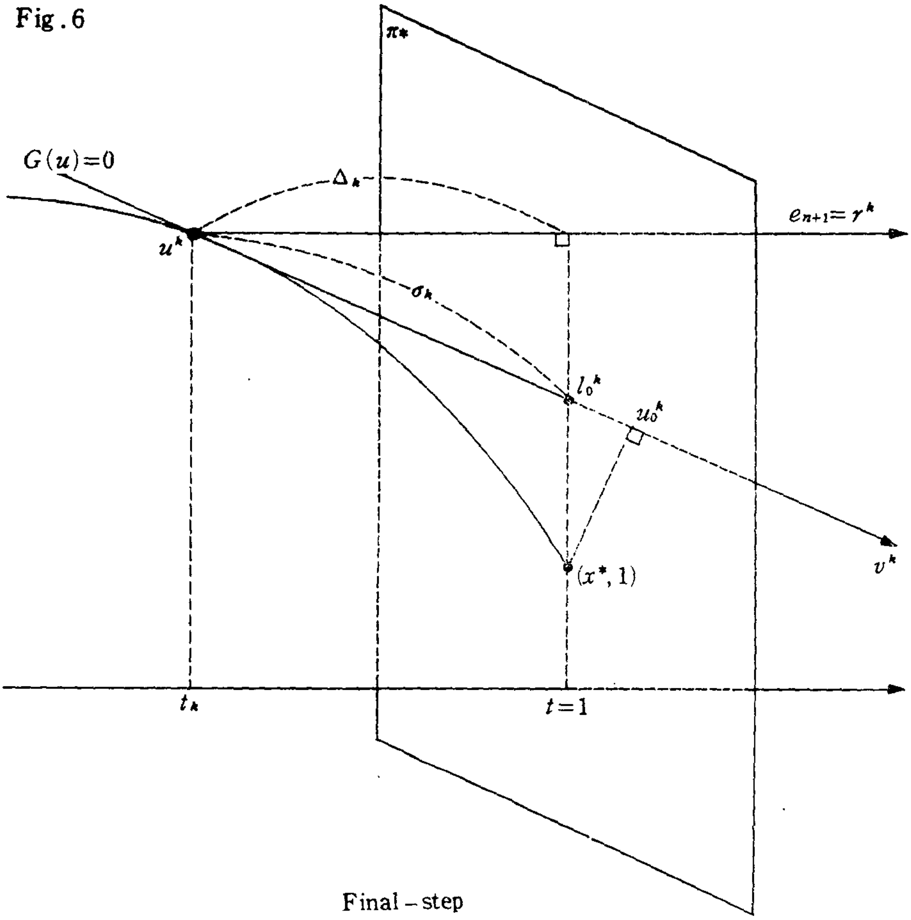


Fig. 5



CONV-TEST-TO-X*

Fig. 6



Final – step

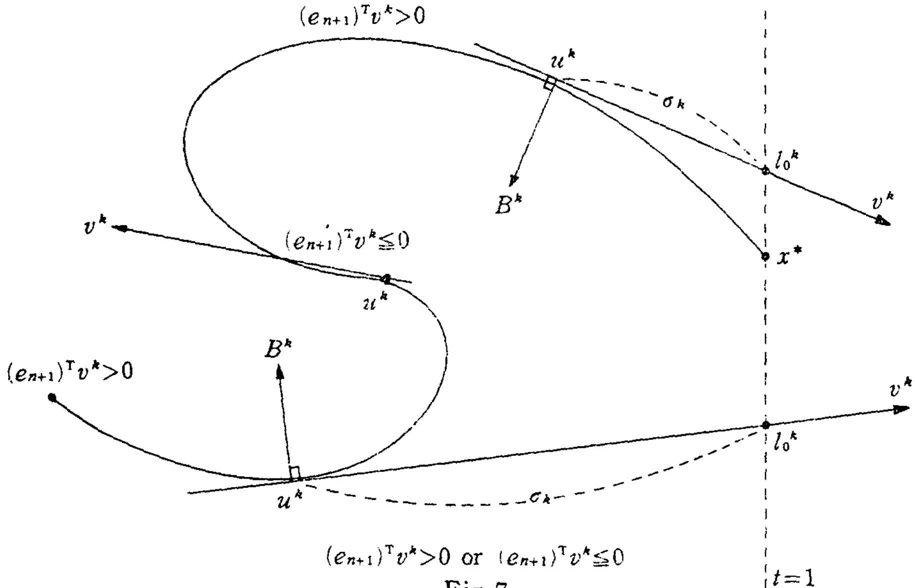


Fig. 7

THEN $x^*; = x_{j+1}$ and STOP
ELSE $j := j+1$; GOTO CORRECT.
ELSE return,

where μ, $\xi \in (0, 1)$ are given numbers.

Final-Step :

See Fig. 6. Recall the hyperplane $\Pi^* = \{u \in R^{n+1} : (e_{n+1})^T u = 1\}$. We compute σ_k from the following equations:

$$(e_{n+1})^T (l_0^k - u^k) = \triangle_k, \quad l_0^k = u^k + \sigma_k v^k.$$

so

$$\sigma_k = \frac{\triangle_k}{(e_{n+1})^T v^k} = \frac{1-t_k}{(e_{n+1})^T v^k}$$

If $\| G(l_0^k) \| \leq \| G(u^k) \| + \varepsilon \sigma_k$, l_0^k is accepted as a approximation for u_{k+1} and because of $t_{k+1} = 1$, may be a approximate root of $F(x) = 0$. Otherwise we call CONV-TEST-TO-X*. If failed we choose $\tau_k \in [\tau_{k-1}, \bar{\tau}]$ and go to ADJUST-STEP.

Adjust-Step :

With the chosen step size or the reduced step size τ_k, u_0^k is computed and t_{k+1} has a new value.

Take-Point :

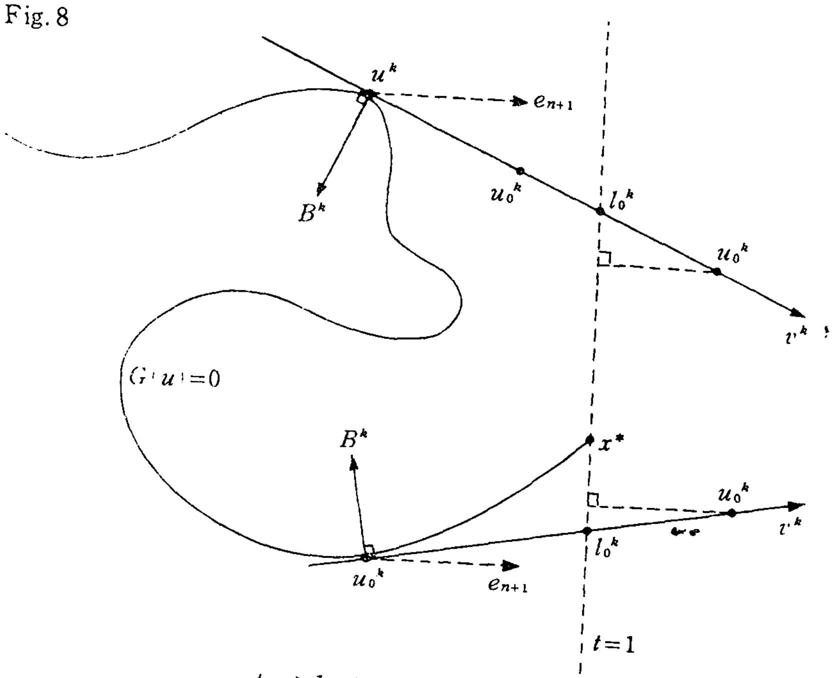
u_0^k is accepted as a successor for u^{k+1}.

The cases $(e_{n+1})^T v^k > 0$ and $(e_{n+1})^T v^k \leq 0$:

$(e_{n+1})^T v^k > 0$: See Fig. 7. We attack Final-Step (see Fig. 6). If failed, we call CONV-TEST-TO-X*. Again if failed, the step size τ_k will be adjusted and the approximate point $u_0^k := u^k + \tau_k v^k$ is computed. The outline for this is:

compute
$$\sigma_k := \frac{1-t_k}{(e_{n+1})^T v^k}$$

Fig. 8



$t_{k+1} > 1$, $t_{k+1} = 1$ or $t_{k+1} < 1$

If $\|G(l_0^k)\| \leq \|G(u^k)\| + \varepsilon\sigma_k$ then $u^{k+1} := l_0^k$ and return, otherwise call CONV-TEST-TO-X*, choose $\tau_k \in [\tau_{k-1}, \bar{\tau}]$ and go to ADJUST-STEP.

$(e_{n+1})^T v^k \leq 0$: Choose $\tau_k \in [\tau_{k-1}, \bar{\tau}]$ and go to ADJUST-STEP.

The cases $t_{k+1} > 1$, $t_{k+1} = 1$ and $t_{k+1} < 1$:

$t_{k+1} > 1$: We compute $u_0^k := u^k + \tau_k v^k$. $0 < \bar{\tau} \leq 1$, $\alpha \in (0,1)$ and call CONV-TEST-TO-X*. If $(v^k)^T e_{n+1} > 0$ then choose $\tau_k \in \min\{\bar{\tau}, \sigma_k\}$ and go to ADJUST-STEP otherwise $\tau_k := \alpha\tau_k$ and go to ADJUST-STEP.

$$t_{k+1} = 1 : \text{If } \|G(l_0^k)\| \leq \|G(u^k)\| + \varepsilon\tau_k$$

Then go to TAKE-POINT otherwise we call CONV-TEST-TO-X*, reduce $\tau_k(\tau_k := \alpha\tau_k)$ and go to ADJUST-STEP.

$t_{k+1} < 1$: We compute $u_1^k := u_0^k - d_k'(u_0^k)^{-1} d_k(u_0^k)$. If $(B^k)^T(u_1^k - u^k) > 0$ and $\|G(u_0^k)\| \leq \|G(u^k)\| + \varepsilon\tau_k$ then go to TAKE-POINT otherwise $\tau_k := \alpha\tau_k$ and go to ADJUST-STEP.

Note: $t_{k+1} := (e_{n+1})^T u_0^k$. see Fig. 8

P.C. method and $\|G(u^{k+1})\| \leq \|G(u^k)\| + \varepsilon\tau_k$ to get u^{k+1} : See Fig. 9.

P.C. method : $\mu \in (0,1)$ is a assigned value and $\tau_k \in [\tau_{k-1}, \bar{\tau}]$ must be chosen for the predictor of u_0^k. τ_k is reduced in order to satisfy some conditions. δ is a preassigned accuracy and it is assumed that $d_k'(u_0^k)$ is regular. To get the approximation u^{k+1} for u_{k+1}, we have the problems for the number of iterates and the accuracy because we don't need the high accuracy for u^{k+1} except for the turning points and our aim is x^*. So the method $\|G(u_0^k)\| \leq \|G(u^k)\| + \varepsilon\tau_k$ will be considered, $\tau_k \in (0,1]$. For the more details see (M1). The algorithm for P.C. can be described as follows:

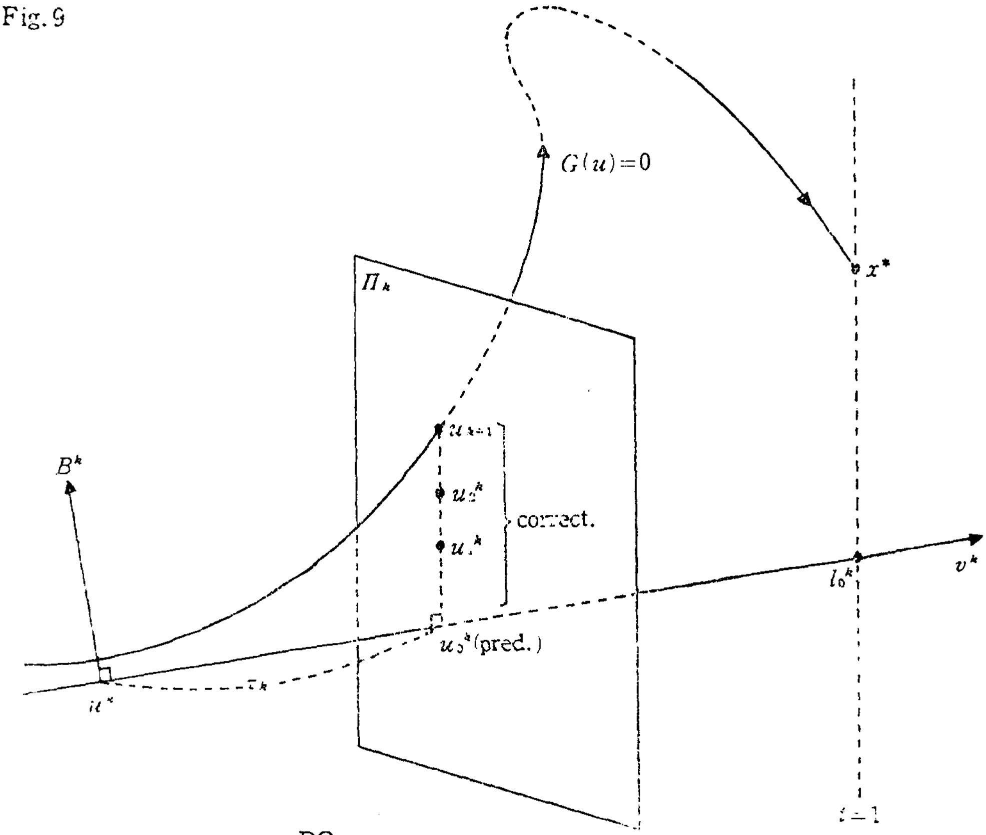
```
PRED.          u_0^k := u^k + τ_k v^k
               i := 0
               u_{i+1}^k := u_i^k − d_k'(u_i^k)^{-1} d_k(u_i^k)
CORRECT.       WHILE (B^k)^T (u_{i+1}^k − u^k) > 0 AND
                     ||G(u_{i+1}^k)|| ≤ μ ||G(u_i^k)||
                     DO IF ||G(u_{i+1}^k)|| ≤ δ
        THEN return (for the next step)
                          ELSE i := i+1
               u_{i+1}^k := u_i^k − d_k'(u_i^k)^{-1} d_k(u_i^k)
               GOTO CORRECT.
               τ_k := ατ_k
               GOTO PRED.
```

The method $\|G(u^{k+1})\| \leq \|G(u^k)\| + \tau_k\varepsilon$: See Fig. 3

Let $\alpha \in (0,1)$ be given, $0 < \tau_{-1} \leq \bar{\tau} \leq 1$ and $\tau_k \in [\tau_{k-1}, \bar{\tau}]$. In (M1) for v^k the generalized incremer function was used. u_0^k is accepted as an appropriate successor of u^k if $\|G(u_0^k)\| \leq \|G(u^k)\| + \varepsilon$ and we try to get the next approximate point. Thus if the parameter $t_N = 1$ is reached, it is see from (M1) that x^* is an $c\varepsilon$-approximation to a root of $F(x) = 0$ where $c > 0$ is a constant. Moreove we need the condition $(B^k)^T(u_1^k - u^k) > 0$ where u_1^k is the first correction to u_0^k.

```
u_0^k := u^k + τ_k v^k
WHILE || G(u_0^k) || > || G(u^k) || + τ_k ε
```

Fig. 9



DO $\tau_k := \alpha\tau_k$

$u_0^k := u^k + \tau_k v^k$

$u^{k+1} := u_0^k$

The failure of the original Menzel's Algorithm and one solution for it : See Fig. 10.

When we keep on moving along zero curve, the failure of Menzel's algorithm could occure in ractice if u_0^k is approached too closely to some unwanted point of zero curve. This failure also vas pointed out in (AG1). Using the curvature vector and the first correction u_1^k for u_0^k, this roblem can be solved but the problem is that we need the second derivatives. So without using he curvature vector we will consider how to get the orthogonal vector B^k with respect to v^k. rom the previous fact there is a full rank neighborhood of Z_0 and by Proposition 3.5, $d_k'(u)$ is egular on the full rank region. Hence we can reduce τ_k until $d_k'(u_0^k)$ is regular. The algorithm s as follows : (a) with curvature vector C^k :

$u_0^k := u^k + \tau_k v^k$

WHILE $d_k'(u_0^k) \neq$ regular

DO $\tau_k := \alpha\tau_k$; $u_0^k := u^k + \tau_k v^k$

$u_1^k := u_0^k - d_k'(u_0^k)^{-1} d_k(u_0^k)$

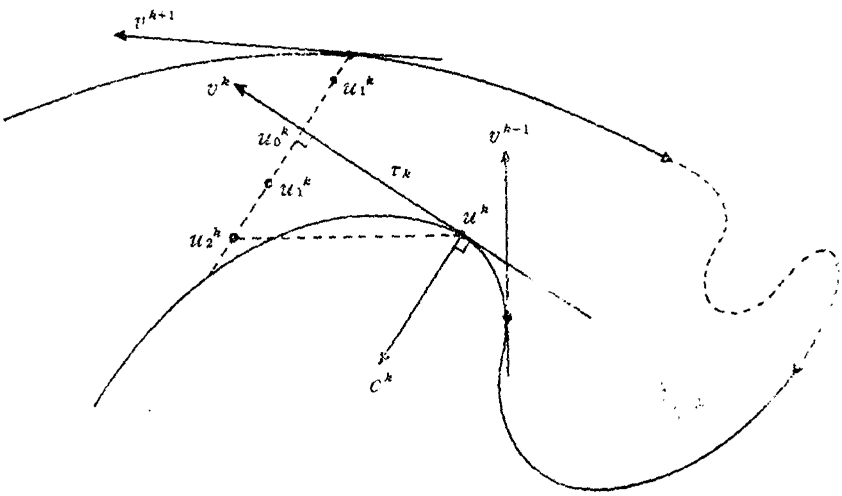
WHILE $(C^k)^T (u_1^k - u^k) < 0$ OR

$\| G(u_0^k) \| > \| G(u^k) \| + \tau_k \varepsilon$

$$\begin{aligned} \text{DO } \tau_k &:= \alpha\tau_k \\ u_0^k &:= u^k + \tau_k v^k \\ u_1^k &:= u_0^k - d_k'(u_0^k)^{-1} d_k(u_0^k) \end{aligned}$$

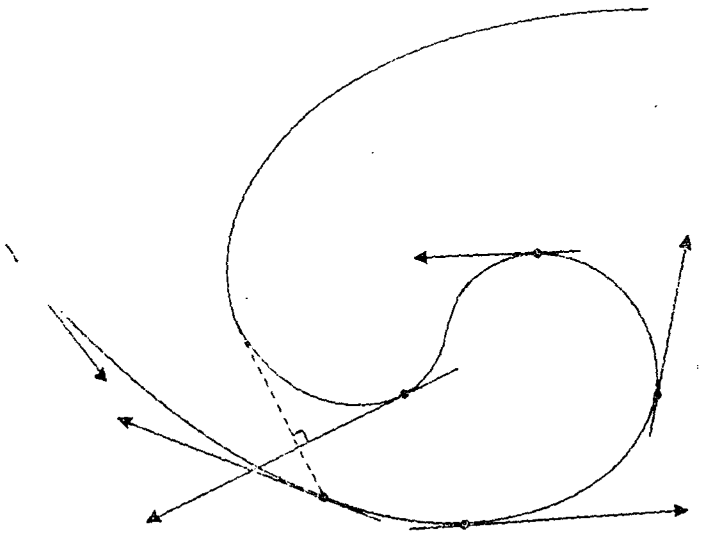
$$u^{k+1} := u_0^k,$$

where C^k is a curvature vector to zero curve Z_0 at u^k. For $d_k'(u^k)$ see Remark of Proposition 3.5.



approaching too closely to some unwanted point of Z_0
Fig. 10

approaching too closely to some unwanted point of Z_0
Fig. 10



returning

'or the better approximation for u^{k+1}, the following codes can be added after WHILE-DO statent.

$u_1^k := u_0^k - d_k'(u_0^k)^{-1} d_k(u_0^k)$

$i := 0$

WHILE $\| G(u_{i+1}^k) \| \leq \| G(u_i^k) \| + \tau_k \varepsilon$

DO $i := i+1$

$u_{i+1}^k := u_i^k - d_k'(u_i^k)^{-1} d_k(u_i^k)$

$m := i$

$u^{k+1} := u_m^k$

ɔ) by finding a vector B^k which is orthogonal to v^k :

ɜe Fig. 4. Suppose a vector B^k which is orthogonal to v^k was found. We consider how to find next vector B^{k+1} and to avoid approaching to some unwanted point of zero curve.

$u_0^k := u^k + \tau_k v^k$

WHILE $d_k'(u_0^k) =$ regular

DO $\tau_k := \alpha \tau_k$; $u_0^k := u^k + \tau_k v^k$

$u_1^k := u_0^k - d_k'(u_0^k)^{-1} d_k(u_0^k)$

WHILE $(B^k)^T (u_1^k - u^k) \leq 0$ OR

$\| G(u_0^k) \| > \| G(u^k) \| + \tau_k \varepsilon$

DO $\tau_k := \alpha \tau_k$

$u_0^k := u^k + \tau_k v^k$

$u_1^k := u_0^k - d_k'(u_0^k) d_k(u_0^k)$

$u^{k+1} := u_0^k$

ıen we compute v^{k+1} with $G'(u^{k+1}) v^{k+1} = 0$, $\| v^{k+1} \| = 1$ and B^{k+1} with $(B^{k+1})^T v^{k+1} = 0$, $\cdot)^T v^k < 0$. If $(B^{k+1})^T v^k > 0$ then $B^{k+1} := -B^{k+1}$.

.3. Parametrizied Homotopy Algorithm.

: $R^n \times R \longrightarrow R^n$ is a weakly singular, C^2-imbedding for F with respect to $a \in R^n$.

OCEDURE MAIN ;

choose $\tau_{-1}, \bar{\tau} \in (0, 1]$ with $0 < \tau_{-1} \leq \bar{\tau}$

choose $\alpha, \beta, \mu, \xi \in (0, 1)$ and $\varepsilon_1 > 0$; $a_1 \in R^n$

$t_0 := 0$; $k := 0$; $i := 1$

$\varepsilon := \varepsilon_i$; (ε-approximation is changed)

$x_0 := a_i$; (start-value is changed)

$u^0 := (x_0, 0)$;

$G(u) := G_{a_i}(u)$;

CALL BASIC TANGENT ALGORITHM

$N_i := N$

$y_0^i := \sum_{j=1}^{n} (e_j)^T u^{N_i}$ (predictor)

$m := 0$

$y_{m+1}^i := y_m^i - F'(y_m^i)^{-1} F(y_m^i)$ (corrector)

(or Runge-Kutta method of order 4)

IF $\| F(y_{m+1}^i) \| \leq \mu \| F(y_m^i) \|$
THEN IF $\| F(y_{m+1}^i) \| \leq \xi$ THEN $x^* := y_{m+1}^i$, STOP
ELSE GOTO (3)

(4) $a_i := y_m^i$
$\varepsilon_i := \beta \varepsilon_i$
$i := i+1$; GOTO (1)

Remark. For corrector of (3) the Runge-Kutta method of order 4 will be favorable. The reas is based on W. Kizner (1964). The procedure is as follows:

$$y_{m+1}^i := y_m^i - F'(y_m^i)^{-1} F(y_m^i)$$

Let $h = -F(y_m^i)$ and $q(y_m^i) = \frac{1}{F'(y_m^i)}$. Then

$$y_{m+1}^i := y_m^i + \frac{1}{6} h \{K_1 + 2K_2 + 2K_3 + K_4\}$$
$$K_1 = q(y_m^i)$$
$$K_2 = q(y_m^i + 1/2\ hK_1)$$
$$K_3 = q(y_m^i + 1/2\ hK_2)$$
$$K_4 = q(y_m^i + hK_3)$$

W. Kizner pointed out the following facts : (a) the order of convergence of the method is 1 (b) by the experience with the method, it does not require as good an initial approximatio Newton's method. (c) The number of correct decimal places is multiplied by five at each because of order 5. (d) A single application of this method would be equivalent to about tw three applications of Newton's method in which the number of correct decimal places is app mately doubled at each iteration. For the more details see (K2).

The Runge-Kutta method of order 4 was also used in Li-York's Algorithm.

PROCEDURE BASIC TANGENT ALGORITHM ;

(0) choose $\alpha \in (0, 1)$; $t_0 := 0$; $k := 0$
(1) (compute v^k and B^k when $k=0$)
compute v^k with $G'(u^k) v^k = 0$, $\| v^k \| = 1$
IF $(v^k)^T e_{n+1} < 0$ THEN $v^k := -v^k$
$B^k := e_{n+1} - r v^k$, $r = (e_{n+1})^T v^k$
IF $(e_{n+1})^T B^k < 0$ THEN $B^k := -B^k$
(2) IF $(e_{n+1})^T v^k > 0$ THEN GOTO (3)
(forward) ELSE GOTO (5)
(3) (final-step)
$\sigma_k := \frac{1 - t_k}{(e_{n+1})^T v^k}$
$l_0^k := u^k + \sigma_k v^k$
IF $\| G(l_0^k) \| \leq \| G(u^k) \| + \varepsilon \sigma_k$
THEN $u_0^k := l_0^k$, GOTO (8)
(4) $u_0^k := l_0^k$

CALL CONV-TEST-TO-X*

(5) choose $\tau_k \in [\tau_{k-1}, \bar{\tau}]$

(6) (adjust-step)

$u_0^k := u^k + \tau_k v^k$

$t_{k+1} := (e_{n+1})^T u_0^k$

(7) CASE ;

$t_{k+1} > 1$: CALL CONV-TEST-TO-X*

IF $(v^k)^T e_{n+1} > 0$

THEN $\tau_k \in \min\{\bar{\tau}, \sigma_k\}$

GOTO (6)

ELSE $\tau_k := \alpha\tau_k$; GOTO (6)

$t_{k+1} = 1$: IF $\| G(u_0^k) \| \leq \| G(u^k) \| + \varepsilon\tau_k$

THEN GOTO (8)

ELSE CALL CONV-TEST-TO-X*

$\tau_k := \alpha\tau_k$; GOTO (6)

$t_{k+1} < 1$: IF $d_k'(u_0^k) \neq$ regular THEN $\tau_k := \alpha\tau_k$ GOTO (6)

compute $u_1^k := u_0^k - d_k'(u_0^k)^{-1} d_k(u_0^k)$

IF $(B^k)^T (u_1^k - u^k) > 0$ AND

$\| G(u_0^k) \| \leq \| G(u^k) \| + \varepsilon\tau_k$

THEN GOTO (8)

ELSE $\tau_k := \alpha\tau_k$; GOTO (6)

(8) (take-point)

$u^{k+1} := u_0^k$; $t_{k+1} := (e_{n+1})^T u^{k+1}$

(9) IF $t_{k+1} = 1$

THEN $N := k+1$; RETURN

ELSE compute v^{k+1} with $G'(u^{k+1}) v^{k+1} = 0$, $\| v^{k+1} \| = 1$

IF $(v^k)^T v^{k+1} < 0$ THEN $v^{k+1} := -v^{k+1}$

compute B^{k+1} with $(B^{k+1})^T v^{k+1} = 0$,

$(B^{k+1})^T v^k < 0$

(10) $k := k+1$; GOTO (2)

Note: For the more convergence discussion of this algorithm see (M1) or (MS1).

5. Increment Functions.

5.1. Euler-Cauchy Increment Function Φ (u, τ).

Here for the derivation the numerical differentiation is used. Also we can use Taylor's series. Let $\alpha_k(\tau)$ be a local parametrization of solutions of

$$\begin{cases} G(u) = G(u^k) \\ (u - u^k)^T v^k = \tau \text{ for } \tau \in [\bar{\tau}-, \bar{\tau}], \end{cases}$$

here $\bar{\tau} > 0$ is a number.

Let $z(u^k, \tau) = \alpha_k(\tau)$. Then

$$G(z(u^k, \tau)) = G(u^k)$$
$$G'(z(u^k, \tau))\partial_2 z(u^k, \tau) = 0, \tag{5.1}$$

where $\partial_2 z(u^k, \tau)$ means the partial derivative to the second variable. On the other hand,

$$(z(u^k, \tau) - u^k)^T v^k = \tau,$$
$$\partial_2 z(u^k, \tau)^T v^k \quad = 1 \tag{5.2}$$

Recall v^k is a vector with $G'(u^k)v^k = 0$, $\| v^k \| = 1$.
From (5.1) and (5.2) let $\tau = 0$. Then we have

$$G'(z(u^k, 0))\partial_2 z(u^k, 0) = 0,$$
$$\partial_2 z(u^k, 0)^T v^k \quad = 1.$$

Hence
$$\partial_2 z(u^k, 0) \quad = v^k.$$

Now

$$\partial_2 z(u^k, 0) = \lim_{\tau \to 0} \frac{z(u^k, \tau) - z(u^k, 0)}{\tau}$$
$$\approx \frac{z(u^k, \tau) - z(u^k, 0)}{\tau}$$

So let
$$\partial_2 z(u^k, 0) = \frac{z(u^k, \tau) - z(u^k, 0)}{\tau}$$

Hence we have $z(u^k, \tau) = z(u^k, 0) + \tau \partial_2 z(u^k, 0)$
$= u^k + \tau v^k$.

Let's define the Euler-Cauchy increment function $\Phi(u, \tau)$ as follows:

$$\Phi(u, \tau) = \begin{cases} \dfrac{z(u^k, \tau) - u^k}{\tau} & : \tau > 0 \\ v^k & : \tau = 0 \end{cases}$$

Note : $\Phi(u, \tau)$ is called the strong consistent increment function in (M1).

5.2. Newton Increment Function $\Phi(u, \tau)$.

Let's consider the following system of equations

$$\begin{cases} G(u) = G(u^k) \\ (v^k)^T (u - u^k) = \tau_k. \end{cases}$$

Let $d_k(u) = \begin{bmatrix} G(u) - G(u^k) \\ (v^k)^T (u - u^k) - \tau_k \end{bmatrix} = 0.$

Then by Proposition 3.5, $d_k'(u) = \begin{bmatrix} G'(u) \\ (v^k)^T \end{bmatrix}$ is regular on $Z(\delta_1)$ and by using the Newton method,

$$u_{i+1} := u_i - d_k'(u_i)^{-1} d_k(u_i), \quad i = 0, 1, 2, \cdots$$

Now

$$d_k'(u_i)^{-1} d_k(u_i) = \left(I - \frac{v_i (v^k)^T}{(v_i)^T v^k}\right) G'(u_i)^+ \{G(u_i) - G(u^k)\},$$

where $G'(u_i)v_i = 0$, $\| v_i \| = 1$.
Let $i = 0$, $z(u^k, \tau) = u_1$ and $u_0 = u^k + \tau v^k$. Then

$$z(u^k, \tau) = u_0 - d_k'(u_0)^{-1} d_k(u_0).$$

Define

$$\Phi(u,\tau)=\begin{cases}\dfrac{z(u^k,\tau)-u^k}{\tau} & : \tau>0\\ v^k & : \tau=0\end{cases}$$

5.3. The Fourth Order Runge-Kutta Increment Function.

From the Euler-Cauchy increment function

$$z(u^k,\tau)=u^k+\tau\partial_2 z(u^k,0).$$

Let $g(u^k,0)=\partial_2 z(u^k,0)=v^k$. Then v^k depends on only $u^k \in z(\delta_1)$.

Let
$$K_1=g(u^k,0)=v^k$$
$$K_2=g\left(u^k+\frac{\tau}{2}K_1,0\right)=v^{k,1},$$
where
$$G'(u^{k,1})v^{k,1}=0,\quad \|v^{k,1}\|=1,$$
$$u^{k,1}=u^k+\frac{\tau}{2}v^k,$$
and let
$$K_3=g\left(u^k+\frac{\tau}{2}K_2,0\right)=v^{k,2},$$
where
$$G'(u^{k,2})v^{k,2}=0,\quad \|v^{k,2}\|=1$$
$$u^{k,2}=u^k+\frac{\tau}{2}K_2$$
and let
$$K_4=g(u^k+\tau K_3,0)=v^{k,3},$$
where
$$G'(u^{k,3})v^{k,3}=0,\quad \|v^{k,3}\|=1$$
$$u^{k,3}=u^k+\tau K_3.$$
Let
$$z(u^k,\tau)=u^k+\tau/6\ (K_1+2K_2+2K_3+K_4).$$
Define
$$\Phi(u,\tau)=\begin{cases}\dfrac{z(u^k,\tau)-u^k}{\tau} & : \tau>0\\ v^k & : \tau=0\end{cases}$$

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