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Ultra Bifilters on an Ordered Set

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——Dedicated to Professor Han Shick Park on his 60th birthday——

I. Introduction

The usual topology on the set R of real numbers is closely related with the order structure on R. Thus it is natural that the theory of topological structures together with the order structures has been an important branch of researches in mathematics. In particular, there has been a growing interest in extensions of topological ordered spaces ([1], [2], [4]). In [4], Y.S. Park has introduced the concept of bifilters with which he has constructed the Wallman type compactification of a topological ordered space and in [3], S.S. Hong has constructed the zero-dimensional ordered compactification with maximal clopen bifilters. More recently, T.H. Choe and S.S. Hong have used the concept of open bifilters on Hausdorff convex ordered spaces to characterize their extensions ([1], [2]).

Our aims to write this paper are to introduce an ultra bifilters on an (partially or quasi-) ordered set and investigate its properties which will be needed for the final biconvergence structure on an ordered set $(X, \leq)$.

For the terminology not introduced in this paper, we refer to [4].

II. Ultra Bifilters

For any set X, $P(X)$ denotes the power set of X, $F(X)$ the set of all filters on X, and $[\mathscr{B}]$ the filter generated by $\mathscr{B}\subset P(X)$.

2. 1. Notation Let $(X, \leq)$ be an ordered set and $A\subset X$.
Then

1) $\uparrow A=\{x\in X \mid a\leq x \text{ for some } a\in A\}$,
2) $\downarrow A=\{x\in X \mid x\leq a \text{ for some } a\in A\}$.

In particular, for a singleton set $A=\{a\}$, let $\uparrow a$ ($\downarrow a$, resp.) denote $\uparrow A$ ($\downarrow A$, resp.).

2. 2. Definition Let $(X, \leq)$ be an ordered set and $A\subset X$.
Then A is said to be an *increasing set* (a *decreasing set*, resp.) if $A=\uparrow A$ ($A=\downarrow A$, resp.).

2. 3. Remark 1) $\uparrow\phi=\phi$ and $\downarrow\phi=\phi$.
2) $A\subset\uparrow A$ and $A\subset\downarrow A$ for any $A\subset X$.

3) If A is an increasing (decreasing, resp.) set, then $X-A$ is a decreasing (increasing, resp.) set for any $A\subset X$.

4) For any family $\{A_i\}_{i\in I}$ of increasing (decreasing, resp.) sets, $\bigcup_{i\in I}A_i$ and $\bigcup_{i\in I}A_i$ are both increasing (decreasing, resp.) sets.

5) $\uparrow(\uparrow A)=\uparrow A$ and $\downarrow(\downarrow A)=\downarrow A$.

The following definition is due to Park [4].

2.4. **Definition** A pair $(\mathscr{F},\mathscr{G})$ of filters on an ordered set $(X,\leq)$ is said to be a *bifilter on* $(X,\leq)$ if it satisfies the following:

1) $\mathscr{F}$ has a base consisting of increasing sets,

2) $\mathscr{G}$ has a base consisting of decreasing sets,

3) for any $F\in\mathscr{F}$ and any $G\in\mathscr{G}$, $F\cap G\neq\phi$.

2.5. **Example** For each $x\in X$, $([\uparrow x], [\downarrow x])$ is a bifilter on $(X,\leq)$.

2.6. **Notation** For any ordered set $(X,\leq)$, let $\mathrm{BiF}(X)$ denote the set of all bifilters on $(X,\leq)$.

Now we introduce the order relation on $\mathrm{BiF}(X)$ as follows:

2.7. **Definition** For $(\mathscr{F},\mathscr{G})$, $(\mathscr{H},\mathscr{K})\in\mathrm{BiF}(X)$, $(\mathscr{F},\mathscr{G})$ is said to be *contained* in $(\mathscr{H},\mathscr{K})$ if $\mathscr{F}\subset\mathscr{H}$ and $\mathscr{G}\subset\mathscr{K}$. In case we write $(\mathscr{F},\mathscr{G})\subseteq(\mathscr{H}, \mathscr{K})$.

It is obvious that the relation $\subseteq$ on $\mathrm{BiF}(X)$ is a partial order, i.e., $(\mathrm{BiF}(X),\subseteq)$ is a poset. Furthermore $(\mathrm{BiF}(X),\subseteq)$ is inductive, i.e., every non-empty chain has an upper bound. Hence by Zorn's lemma, $\mathrm{BiF}(X)$ has a maximal element, for any $X\neq\phi$. The maximal bifilter is called an *ultra bifilter* on $(X,\leq)$.

2.8. **Proposition** *For any* $x\in X$, $([\uparrow x], [\downarrow x])$ *is an ultra bifilter on* $(X,\leq)$.

Proof. Suppose that there is a bifilter $(\mathscr{F},\mathscr{G})$ such that $([\uparrow x], [\downarrow x])\subseteq(\mathscr{F},\mathscr{G})$. Let $F\in\mathscr{F}$ and $G\in\mathscr{G}$, then there exists $A=\uparrow A\in\mathscr{F}$ and $B=\downarrow B\in\mathscr{G}$ such that $A\subset F$ and $B\subset G$. Since $(\mathscr{F},\mathscr{G})\in\mathrm{BiF}(X)$, there exists $A\cap\uparrow x\in\mathscr{F}$ and $B\cap\downarrow x\in\mathscr{G}$ such that $(A\cap\uparrow x)\cap(B\cap\downarrow x)\neq\phi$. Take any element $t\in(A\cap\uparrow x)\cap(B\cap\downarrow x)$. Since $t\in\uparrow x$ and $t\in\downarrow x$, $x\leq t$ and $t\leq x$. Thus $y\in\uparrow x$, i.e. $x\leq y$ imply $t\leq y$ since $t\leq x$. Since $t\in A$, $y\in A$. Hence $\uparrow x\subset A\subset F$. i.e., $F\in[\uparrow x]$. Similarly if $y\in\downarrow x$, then $y\in B$. Hence $\downarrow x\subset B\subset G$. i.e., $G\in[\downarrow x]$. Thus $(\mathscr{F},\mathscr{G})\subseteq([\uparrow x], [\downarrow x])$. This completes the proof.

2.9. **Remark** 1) Every bifilter on $(X,\leq)$ is contained in an ultra bifilter.

2) For $(\mathscr{F},\mathscr{G})$, $(\mathscr{H},\mathscr{K})\in\mathrm{BiF}(X)$, $(\mathscr{F}\cap\mathscr{H}, \mathscr{G}\cap\mathscr{K})$ is again a bifilter on $(X,\leq)$.

2.10. **Theorem** *Let* $(F,\mathscr{G})$ *be a bifilter on* $(X,\leq)$.

Then the followings are equivalent:

1) $(\mathscr{F},\mathscr{G})$ *is an ultra bifilter.*

2) *For any increasing set* U, $U\in\mathscr{F}$ *if and only if* $U\cap F\cap G\neq\phi$ *for any* $F\in\mathscr{F}$ *and* $G\in\mathscr{G}$ *and for any decreasing set* V, $V\in\mathscr{G}$ *if and only if* $V\cap F\cap G\neq\phi$ *for any* $F\in\mathscr{F}$ *and* $G\in\mathscr{G}$.

Proof. 1)$\Longrightarrow$2) If U is an increasing set with $U\in\mathscr{F}$, $U\cap F\in\mathscr{F}$ for any $F\in\mathscr{F}$ and hence $U\cap F\cap G\neq\phi$ for any $G\in\mathscr{G}$. Similarly, for any decreasing set V if $V\in\mathscr{G}$, then $V\cap F\cap G\neq\phi$ f any $F\in\mathcal{F}$ and $G\in\mathcal{G}$. Conversely, assume that U is an increasing set such that $U\cap F\cap G\neq\phi$ for any $F\in\mathcal{F}$ and $G\in\mathcal{G}$. Let the family $\{B_i|i\in I\}$ of increasing sets indexed by a class I be a base for $\mathcal{F}$. Then $U\cap B_i\neq\phi$ for each $i\in I$, by hypothesis. Thus the family $\mathcal{B}_0=\{U\cap B_i|i\in I\}$ of the increasing sets generates a filter $\mathcal{F}_0$. Since $(U\cap B_i)\cap G\neq\phi$ for each $i\in I$ and $G\in\mathcal{G}$, $(\mathcal{F}_0,\mathcal{G})$ is a bifilter on $(X,\leq)$. Clearly, $\mathcal{F}\subset\mathcal{F}_0$ and hence $(\mathcal{F},\mathcal{G})\subseteq(\mathcal{F}_0,\mathcal{G})$. Since $(\mathcal{F},\mathcal{G})$ is an ultra bifilter $(\mathcal{F},\mathcal{G})=(\mathcal{F}_0,\mathcal{G})$, so that $U\in\mathcal{F}$. Similarly, if V is a decreasing set such that $V\cap F\cap G\neq\phi$ for any $F\in\mathcal{F}$ and $G\in\mathcal{G}$, then $V\in\mathcal{G}$.

2) $\Longrightarrow$1) Suppose that there is a bifilter $(\mathcal{H},\mathcal{K})$ such that $(\mathcal{F},\mathcal{G})\subseteq(\mathcal{H},\mathcal{K})$. Let $A\in\mathcal{H}$, then there exists an increasing set U such that $U\subset A$. Suppose $U\notin\mathcal{F}$. By the condition 2), there exists $F\in\mathcal{F}$ and $G\in\mathcal{G}$ such that $U\cap F\cap G=\phi$. But $U\cap F\in\mathcal{H}$ and $G\in\mathcal{K}$ since $\mathcal{F}\subset\mathcal{H}$ and $\mathcal{G}\subset\mathcal{K}$. Hence $(U\cap F)\cap G\neq\phi$, which is a contradiction to $U\cap F\cap G=\phi$. Hence $U\in\mathcal{F}$. Thus $A\in\mathcal{F}$. Similarly, if $B\in\mathcal{K}$, then $B\in\mathcal{G}$. This completes the proof.

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