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Certain Class of Regular Univalent Functions

by Suk-Young Lee

Ewha Womans University, Seoul, Korea

— Dedicated to Professor Han Shick Park on his 60th birthday —

1. Introduction

The class S of functions $f(z)$ which are regular and univalent in the open unit disk $\triangle=\{z : |z|<1\}$ each normalized by the conditions

$$f(0)=0 \text{ and } f'(0)=1 \tag{1.1}$$

as been studied intensively for about seventy years. A large and very successful portion of this ork has dealt with subclasses of S characterized by some geometric property of $f[\triangle]$, the image f $\triangle$ under $f(z)$, which is expressible in analytic terms. The class of starlike functions in S is ne of these [5] ; $f(z)$ is starlike with respect to the origin if the segment $[0,f(z)]$ is in $f[\triangle]$ or every z in $\triangle$ and this condition is equivalent to requiring that

$$zf'(z)/f(z) \tag{1.2}$$

ave a positive real part in $\triangle$.

In this note the class of "disk-like" functions is introduced by placing restrictions on the behavr of the imaginary part of (1.2) and a representation formula for these functions is given in rms of Robertson's funtions which are starlike in one direction [6].

2. The class of disk-like functions

In the definition which follows, the notation $g(t)\in\uparrow(a,b)$ means that $g(t)$ is strictly increasing i the interval $a<t<b$. Similarly $g(t)\in\downarrow(a,b)$ means that $g(t)$ is strictly decreasing in the inrval $a<t<b$.

Definition 1. $f(z)$ is regular in $\triangle$, satisfies (1.1) and $f(z)\neq 0$ for z in $\triangle$ unless $z=0$. $f(z)$ *disk-like* with respect to the origin in $\triangle$, or $f(z)\in F$, if and only if one of the following conitions is satisfied:

(i) There exists a constant $\rho=\rho(f(z))>0$ and two functions

$$\theta_k(r)=\theta_k(r\,;f(z)),\ k=1,2\,;\ \ 0<\theta_2(r)-\theta_1(r)<2\pi\,;$$

ich that for $\rho<r<1$

$$|f(re^{i\theta})|\in\downarrow(\theta_1(r),\theta_2(r)) \text{ and } |f(re^{i\theta})|\in\uparrow(\theta_2(r),\theta_1(r)+2\pi). \tag{2.1}$$

(ii) $f(z)$ is regular in $\overline{\triangle}$, the closure of $\triangle$, and there exist real numbers θ_1 and θ_2, $<\theta_2-\theta_1<2\pi$, such that

$$|f(e^{i\theta})| \in \downarrow (\theta_1, \theta_2) \text{ and } |f(e^{i\theta})| \in \uparrow (\theta_2, \theta_1+2\pi). \tag{2.2}$$

Functions in F are not necessarily univalent; F' will denote the class of univalent members of F, i.e., $F'=F\cap S$. $\theta_1(r)$ may be chosen so that $0\leq\theta_1(r)<2\pi$ and a similar choice can be made for θ_1 in (ii). Hereafter, unless otherwise implied, $f(z)$ in F will satisfy part (i) of the definition as this is no restriction on the development.

A subclass of F, the *circularly symmetric functions* was introduced earlier by Jenkins [3] and has appeared in the recent investigations of Krzyz and Reade [4]. Tammi [8] has obtained distortion theorems and coefficient bounds for functions defined in terms of restrictions on the quotient (1.2).

A geometric interpretation of the conditions of Definition 1 may be given. Let $f(z)$, $\rho, \theta_1(r)$ and $\theta_2(r)$ satisfy (i), let $|f(re^{i\theta_1(r)})|=R_1(r)$ and $|f(re^{i\theta_2(r)})|=R_2(r)$ and let C_r, the image of $|z|=r$ under $f(z)$, enclose a domain D_r. Then C_r is contained entirely in the annulus $R_2(r)\leq|w|\leq R_1(r)$ and C_r intersects every circle $|w|=R$, $R_2(r)<R<R_1(r)$, exactly twice (perhaps in the same point) for $\rho<r<1$. If for $R_2(r)<R<R_1(r)$ we let $\Phi_1(r\,;R)$ and $\Phi_2(r\,;R)$ be the arguments, chosen to be unique by continuity, of the intersection of $|w|=R$ with the arcs $\{f(re^{i\theta})\,|\,\theta_1(r)<\theta<\theta_2(r)\}$ and $\{f(re^{i\theta})\,|\,\theta_2(r)<\theta<\theta_1(r)+2\pi\}$, respectively, then D_r contains the arc $Re^{i\Phi}$,

$\Phi_2(r\,;R)<\Phi<\Phi_1(r\,;R)$. It is clear that $f(z)$ is not univalent in $\triangle$ if there exist r and R such that $\Phi_1(r\,;R)-\Phi_2(r\,;R)\geq 2\pi$ for all choices of arguments. On the other hand $f(z)$ is univalent in $\triangle$ whenever $0<\Phi_1(r\,;R)-\Phi_2(r\,;R)<2\pi$ for all admissible r and R and appropriate $\Phi_1(r\,;R)$ and $\Phi_2(r\,;R)$, and conversely; hence $f(z)\in F'$, or $f(z)=z$, if and only if every circle centered at the origin meets $f[\triangle]$ in a single, non-overlapping arc or not at all. This gives rise to the following observation which we will use; $f(z)$ is univalent for $|z|\leq r$ if the plane can be cut from $f(re^{i\theta_2(r)})$ to ∞ by a curve which does not meet C_r in any point other than $f(re^{i\theta_2(r)})$.

Returning now to the definition, we see that for $\rho<r<1$, (2.1) can be written as

$$\frac{\partial}{\partial\theta}\Big|f(re^{i\theta})\Big| \begin{cases} <0 & \text{for } \theta_1(r)<\theta<\theta_2(r) \\ >0 & \text{for } \theta_2(r)<\theta<\theta_1(r)+2\pi, \end{cases} \tag{2.3}$$

or as

$$\frac{\partial}{\partial\theta}\log\Big|f(re^{i\theta})\Big| \begin{cases} <0 & \text{for } \theta_1(r)<\theta<\theta_2(r), \\ >0 & \text{for } \theta_2(r)<\theta<\theta_1(r)+2\pi. \end{cases} \tag{2.4}$$

Because

$$Re\left\{\frac{\partial}{\partial\theta}\log f(z)\right\}=Re\left\{\frac{d}{dz}\log f(z)\cdot\frac{dz}{d\theta}\right\}=Re\left\{i\frac{zf'(z)}{f(z)}\right\}=-Im\left\{\frac{zf'(z)}{f(z)}\right\}$$

for $z=re^{i\theta}$, (2.2) is equivalent to

$$Im\left\{\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}\right\} \begin{cases} >0 & \text{if } \theta_1(r)<\theta<\theta_2(r) \\ <0 & \text{if } \theta_2(r)<\theta<\theta_1(r)+2\pi. \end{cases} \tag{2.5}$$

The last form upon normalization relates to functions starlike in one direction. Letting

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k, \quad z\in\triangle \tag{2.6}$$

then

$$\frac{zf'(z)}{f(z)}-1=a_2z-(2a_3-a_2^2)z^2+\cdots \tag{2.7}$$

onsequently $a_2 \neq 0$, otherwise (2.7) has a multiple zero which is a contradiction of (2.5). Therere it is no restriction to assume that a_2 is real and positive. The above discussion yields the llowing conclusions.

Theorem 1. *$f(z)$ has the form* (2.6) *and $a_2>0$. $f(z)$ is in F if and only if*

$$a_2^{-1}\left\{\frac{zf'(z)}{f(z)}-1\right\}$$

starlike in the direction of the real axis.

It should be noticed that if $a_2 \neq 0$, then $f(z)$ is not odd; this is consistent with the geometrical terpretation given above. Consequently the identity function $f(z)=z$ is not in F. A modification Definition 1 to admit simple monotonic rather than strictly monotonic functions in (2.1) and (2.2) mits the identity function into F, in which case (2.7) is identically zero.

Making use of the fact that $g(z)$ regular in $\triangle$ and normalized by (1.1) is convex in one rection if and only if $zg'(z)$ is starlike in one direction [6] we can write the last theorem in other form.

Corollary. *$f(z)=z+\sum_{k=2}^{\infty}a_k z^k$, $a_2>0$, is in F if and only if $f(z)=ze^{a_2 g(z)}$ and $g(z)$ is convex in e direction of the imaginary axis.*

This can be obtained directly from Theorem 1 or by observing that for $z=re^{i\theta}$, $\rho<r<1$, (2.4) equivalent to

$$Re\frac{\partial}{\partial\theta}\log\left(\frac{f(z)}{z}\right)\begin{cases}<0 & \text{for } \theta_1(r)<\theta<\theta_2(r),\\ >0 & \text{for } \theta_2(r)<\theta<\theta_1(r)+2\pi.\end{cases} \tag{2.8}$$

nce,

$$\log\left(\frac{f(z)}{z}\right)=a_2 z+\left(a_3-\frac{a_2^2}{2}\right)z^2+\cdots$$

convex in one direction.

3. A Univalent Subclass of F

or a fixed ϕ, $-\pi<\phi\leq\pi$, we say that $f(z)$ regular in $\triangle$, or in $\overline{\triangle}$, and normalized by (1.1) n $\mathcal{F}_\phi$ if and only if $f(z)$ is starlike in the direction with inclination ϕ. That is, the line $=te^{i\phi}$, t real, intersects C_r, the image of $|z|=r$ under $f(z)$, for r near or equal to 1 exactly ce. Evidently $f(z)\in\mathcal{F}_\phi$ whenever $e^{-i\phi}f(ze^{i\phi})$ is starlike in the direction of the real axis, and versely. Consequently, there exist functions $\tau_1(r;f(z))$ and $\tau_2(r;f(z))$ such that for $\sigma<r<1$, for $r=1$, and suitable choice of arguments, $\phi<\arg\{f(re^{i\phi})\}<\phi+\pi$ whenever

$$\tau_1(r;f(z))<\theta<\tau_2(r;f(z))$$

$$\phi+\pi<\arg\{f(re^{i\phi})\}<\phi+2\pi$$

enever

$$\tau_2(r;f(z))<\theta<\tau_1(r;f(z))+2\pi.$$

arly σ depends on $f(z)$, $\sigma=\sigma(f(z))$, and in the case $r=1$ it is assumed that $f(z)$ is holomor- in $\overline{\triangle}$.

sing these ideas we may restate Theorem 1 in the following useful form.

Theorem 2. *$f(z)$ has series representation (2.6) and $a_2 \neq 0$. $f(z)$ is in F if and only if*

$$a_2^{-1}\left\{\frac{zf'(z)}{f(z)} - 1\right\} \in \mathcal{F}_\phi$$

for $\phi = \operatorname{Arg}\{a_2^{-1}\}$.

Choose $\alpha = \operatorname{Arg} a_2$, where Arg denotes the principal argument, then

$$f_0(z) = e^{i\alpha} f(ze^{-i\alpha}) = z + |a_2| z^2 + \cdots$$

is in F, since membership in F is preserved under rotation. Therefore writing

$$|a_2|^{-1}\left\{\frac{zf_0'(z)}{f_0(z)} - 1\right\} = g(z),$$

where $g(z)$ is starlike in the direction of the real axis, yields, upon substitution, the relation

$$a_2^{-1}\left\{\frac{zf'(z)}{f(z)} - 1\right\} = e^{i\alpha} g(ze^{i\alpha}).$$

The last function is in $\mathcal{F}_{-\alpha}$ and $-\alpha = \phi$.

Definition 2. $f(z)$ and $g(z)$, both in $\mathcal{F}_\phi$, are *similar* if and only if

$$\tau_k(r\,; f(z)) = \tau_k(r\,; g(z)), \quad k = 1, 2$$

for $r = 1$ when $f(z)$ and $g(z)$ are regular in $\overline{\triangle}$ and for $\sigma < r < 1$,

$$\sigma = \sigma(f(z)\,; g(z)),$$

otherwise.

Let $\tau_k(r) = \tau_k(r\,; f(z)) = \tau_k(r\,; g(z))$, $k = 1, 2$, then a geometric interpretation of similarity is that for r near or equal to 1, $f(re^{i\tau_1(r)})$ and $g(re^{i\tau_1(r)})$ both lie on one ray of the line $w = te^{i\phi}$, t real, whereas $f(re^{i\tau_2(r)})$ and $g(re^{i\tau_2(r)})$ both lie on the complementary ray. $\mathcal{F}_0$ is the class of functions starlike in the direction of the real axis and any two typically-real functions, all of which are in $\mathcal{F}_0$, are similar.

Theorem 3. *If*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad a_2 \neq 0, \quad \textit{and} \quad g(z) = a_2^{-1}\left\{\frac{zf'(z)}{f(z)} - 1\right\}$$

are similar and in $\mathcal{F}_\phi$, $\phi = \operatorname{Arg}\{a_2^{-1}\}$, then $f(z)$ is univalent in $\triangle$.

To give a proof let $\tau_k(r)$, $k = 1, 2$, and σ be as in Definition 2 and the above paragraph. Suppose furthermore that $f(re^{i\tau_1(r)})$ and $g(re^{i\tau_1(r)})$ fall on the ray $w = te^{i\phi}$, $t > 0$, for $\sigma < r < 1$. Then for $z = re^{i\phi}$, $\sigma < r < 1$ and appropriate choice of arguments

$$\phi < \arg\{g(z)\} < \phi + \pi \quad \text{for } \tau_1(r) < \theta < \tau_2(r)$$

and

$$\phi + \pi < \arg\{g(z)\} < \phi + 2\pi \quad \text{for } \tau_2(r) < \theta < \tau_1(r) + 2\pi$$

or

$$0 < \arg\{e^{-i\phi} g(z)\} < \pi \quad \text{when } \tau_1(r) < \theta < \tau_2(r)$$

and

$$\pi < \arg\{e^{-i\phi} g(z)\} < 2\pi \quad \text{when } \tau_2(r) < \theta < \tau_1(r) + 2\pi.$$

This is equivalent to

$$Im\left\{\frac{zf'(z)}{f(z)}\right\} \begin{cases} > 0, & \tau_1(r) < \theta < \tau_2(r) \\ < 0, & \tau_2(r) < \theta < \tau_1(r) + 2\pi. \end{cases}$$

Consequently, $|f(re^{i\phi})|$ is strictly decreasing for $\tau_1(r) < \theta < \tau_2(r)$ and for a fixed r, $\rho < r < 1$, hence

$(re^{i\phi})$ cuts every circle $|w|=R$, $|f(re^{ir_2(r)})|<R<|f(re^{ir_1(r)})|$ exactly once at a point $Re^{i\mu}$, $<\mu<\phi+\pi$ because $f(z)\in\mathcal{F}_\phi$. We see in the same way that every semicircle $Re^{i\mu}$, $+\pi<\mu<\phi+2\pi$ is intercepted only once. Therefore $f(z)$ is univalent on $|z|=r$ and is, for that reason [5], univalent for $|z|\leq r$. A similar argument covers the remaining cases.

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