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Equivalent Formulations of Zorn's Lemma and Other Maximum Principles

by Sehie Park

Seoul National University, Seoul, Korea

— Dedicated to Professor Han Shick Park on his 60th birthday —

Abstract. In this paper, we give a result that maximum principles including Zorn's lemma can be regarded as various types of fixed point theorems. Our main application is that the well-known ordering principles in nonlinear analysis including the Bishop-Phelps argument and a number of its generalizations can be converted to fixed point theorems and vice versa. Consequently, we obtain new results and unify many known results.

Recently, there have been appeared a number of ordering principles in nonlinear analysis including the well-known Bishop-Phelps lemma [4] and its extensions given by Phelps [22], Ekeland [13], Brøndsted [8], Brézis-Browder [7], Altman [3], Turinici [26], [28], and Kang-Park [15]. It is well-known that any maximum principle including Zorn's lemma implies a fixed point result on expanding maps f (that is, $x \leq fx$ for all x).

In this paper, we begin with a metatheorem on the equivalency of maximum principles and various types of fixed point results. We apply this to Zorn's lemma and some useful forms of the above-mentioned principles. Consequently, we obtain some new results and unify a number of known results.

Let 2^X denote the power set of a set X, and $\sim$ the negation.

In the Zermelo-Fraenkel set theory with Axiom of Choice, we have the following.

Theorem 1. *Let X be a set, A its nonempty subset, and $G(x,y)$ a sentence formula for $x, y \in X$. Then the following are equivalent:*

(i) *There exists an element $v \in A$ such that $G(v,w)$ for any $w \in X \setminus \{v\}$.*

(ii) *If $T: A \longrightarrow 2^X$ is a multimap such that for any $x \in A \setminus T(x)$, there exists a $y \in X \setminus \{x\}$ satisfying $\sim G(x,y)$, then T has a fixed element $v \in A$, that is, $v \in T(v)$.*

(iii) *If $f: A \longrightarrow X$ is a map such that for any $x \in A$ with $x \neq fx$, there exists a $y \in X \setminus \{x\}$ satisfying $\sim G(x,y)$, then f has a fixed element $v \in A$, that is, $v = fv$.*

(iv) *If $T: A \longrightarrow 2^X \setminus \{\phi\}$ is a multimap such that $\sim G(x,y)$ holds for any $x \in A$ and any $y \in T(x) \setminus \{x\}$, then T has a stationary element $v \in A$, that is, $\{v\} = T(v)$.*

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(v) *If* $\mathcal{F}$ *is a family of maps* $f: A \longrightarrow X$ *satisfying* $\sim G(x,fx)$ *for all* $x \in A$ *with* $x \neq fx$, *then* $\mathcal{F}$ *has a common fixed element* $v \in A$, *that is,* $v = fv$ *for all* $f \in \mathcal{F}$.

Proof. See [22].

A nonempty poset (partially ordered set) is said to be *inductively ordered* if

(A) every nonempty simply ordered subset has an upper bound.

We adopt the following form of Zorn's lemma.

Theorem. *Let* $(P, \leq)$ *be an inductively ordered set. Then for any* $a \in P$, *there exists* ɩ *maximal element* $v \in P$ *such that* $v \in S(a) = \{x \in P \mid a \leq x\}$.

An equivalent form of Zorn's lemma is obtained by replacing (A) by the following ([2], [5], [6]) :

(B) every nonempty well-ordered subset has an upper bound.

As the first application of Theorem 1, we have the following:

Theorem 2. *Let* $(P, \leq)$ *be a poset,* $a \in P$, *and* $S(a) = \{x \in P \mid a \leq x\}$ *satisfy* (A) *or* (B). *The we have the following equivalent conditions hold:*

(i) *There exists a* $v \in S(a)$ *such that* $\sim(v \leq w)$ *for any* $w \in P \setminus \{v\}$.

(ii) *If* $T: S(a) \longrightarrow 2^P$ *is a multimap such that*

$$\forall x \in S(a) \setminus T(x),\ \exists y \in P \setminus \{x\} \text{ such that } x \leq y,$$

then T *has a fixed element* $v \in S(a)$.

(iii) *If* $f: S(a) \longrightarrow P$ *is a map such that* $x \leq fx$ *for any* $x \in S(a)$, *then* f *has a fixed elemer* $v \in S(a)$.

(iv) *If* $T: S(a) \longrightarrow 2^P \setminus \{\phi\}$ *is a multimap such that*

$$\forall x \in S(a),\ \forall y \in T(x),\ x \leq y \text{ holds,}$$

then T *has a stationary element* $v \in S(a)$.

(v) *If* $\mathcal{F}$ *is a family of maps* $f: S(a) \longrightarrow P$ *satisfying* $x \leq fx$ *for all* $x \in S(a)$, *then* $\mathcal{F}$ *has common fixed element* $v \in S(a)$.

Note that (i) is equivalent to Zorn's lemma and (iii) due to Abian [1], Kneser [17] and Morc anu [19] with constructive proofs.

Actually, (i)$\Longrightarrow$(iii) is a simple observation([9]). (iii)$\Longrightarrow$(i) is given in [1], [2], [19]. Kasaha [16] obtained (i)$\Longrightarrow$(v). Also note that Proposition 1.6 of [24] is a consequence of (i)$\Longrightarrow$(iii)

A slightly different version of Theorem 2 can be stated as follows:

Theorem 3. *Let* $(P, \leq)$ *be a poset and* $A \subset P$ *a nonempty subset. Then the followings* ɩ *equivalent.*

(i) *A has a maximal (minimal) element.*

(ii) *If* $T: A \longrightarrow 2^P$ *is a multimap such that*

$$\forall x \in A \setminus T(x),\ \exists y \in A \text{ such that } x < y\ (y < x),$$

then T *has a fixed element.*

(iii) *If* $f: A \longrightarrow P$ *is a map such that*

$$\forall x \in A,\ x \neq fx \Longrightarrow x < fx\ (fx < x) \text{ and } fx \in A,$$

then f *has a fixed element.*

(iv) *If* $T: A \longrightarrow 2^P \setminus \{\phi\}$ *is a multimap such that*

$$\forall x \in A, \ \forall y \in T(x) \setminus \{x\}, \ x < y \ (y < x) \text{ and } y \in A,$$

then T *has a stationary element.*

(v) *If* $\mathcal{F}$ *is a family of maps* $f: A \longrightarrow P$ *satisfying*

$$\forall x \in A, \ x \neq fx \Longrightarrow x < fx \ (fx < x) \text{ and } fx \in A,$$

then $\mathcal{F}$ *has a common fixed element.*

Let $(P, \leq)$ be a poset and $f: P \longrightarrow P$ a selfmap. Then f is said to be ***isotone*** if $f(x) \leq f(y)$ whenever $x \leq y$.

Corollary 1. (Smithson [25]). *Let* $(P, \leq)$ *be a poset. Let* $e \in P$ *and* $\mathcal{F}$ *be a commuting family of isotone maps of* P *(i.e.,* $f \circ g \equiv g \circ f$ *for all* $f, g \in \mathcal{F}$*). Suppose that* $e \leq fe$ *for all* $f \in \mathcal{F}$ *and every chain containing* e *has a least upper bound in* P*, then* $\mathcal{F}$ *has a common fixed element.*

Proof. Let $A = \{x \in P \mid e \leq x \text{ and } x \leq fx \text{ for all } f \in \mathcal{F}\}$. Then $e \in A$ and $A \neq \phi$. Let C be a chain in A and $x_0 = \text{lub } C$. Then for any $f \in \mathcal{F}$ and any $x \in C$, $x \leq x_0$ implies $x \leq fx \leq fx_0$. So fx_0 is an upper bound for C, whence $x_0 \leq fx_0$. That is, $x_0 \in A$. By Zorn's lemma, A has a maximal element.

Let $x \in A$ and $f \in \mathcal{F}$. If $x \neq fx$, $e \in x \setminus fx$. And for each $g \in \mathcal{F}$, $x < gx$ implies $g(fx) = f(gx) \leq f(x)$. Thus $fx \in A$. By (v) of Theorem 3, $\mathcal{F}$ has a common fixed element.

Corollary 1 generalizes Knaster-Tarski's theorem (Theorem I. 4. 1. in [11]), which also can be derived from (i)$\Longleftrightarrow$(iii) of Theorem 3. Also note that De Marr's theorem[11] is a dual form of Corollary 1.

Corollary 2. (Höft and Höft [14]). *Let* $(P, \leq)$ *be a poset. Suppose that every nonempty chain in* P *has a l.u.b. and a g.l.b. in* P*. If* $f: P \longrightarrow P$ *is an isotone map and there is an* $e \in P$ *such that* e *and* fe *are comparable, then* f *has a fixed element.*

Proof. This is an easy consequence of (i)$\Longleftrightarrow$(iii) of Theorem 3, by setting

$A = \{x \in P \mid x \leq fx \text{ and } e \leq x\}$ if $e \leq fe$ or setting $A = \{x \in P \mid fx \leq x \text{ and } x \leq e\}$ if $fe \leq e$.

A number of earlier fixed point results on posets may follow from Theorem 2 or Theorem 3.

In nonlinear analysis, there have appeared a number of constructive maximum principles. In fact, a set $(X, \leq)$ with a quasi-order $\leq$ (that is, reflexive and transitive) has some additional structure like a metric space, we have the following application of Theorem 1, which can be regarded a constructive "countably inductive" version of Zorn's lemma.

Theorem 4. *Let* $(X, d, \leq)$ *be a quasi-ordered metric space, and* $a \in X$ *such that*

(1) *any nondecreasing Cauchy sequence in* $S(a)$ *has an upper bound, and*

(2) *for any* $x \in S(a)$ *and* $\varepsilon > 0$*, there exists* $y \in S(x)$ *such that diam* $S(y) < \varepsilon$.

Then (i)—(v) *of Theorem* 2 *hold.*

Proof. (i) By (2), since $a \in S(a)$, there exists a $y_1 \in S(a)$ such that diam $S(y_1) < 1$. Suppose $y_1, \cdots, y_n \in S(a)$ are chosen. Then there exists $y_{n+1} \in S(y_n)$ such that diam $S(y_{n+1}) < 1/(n+1)$. By induction, we obtain a nondecreasing Cauchy sequence $\{y_n\}$. By (1), there exists a $v \in S(a)$ such that $y_n \leq v$ for all n. Since $v \in S(y_n)$ and diam $S(y_n) \to 0$ as $n \to \infty$, we have $y_n \to v$. Since $v \leq z$ implies $v, z \in S(y_n)$, we have $d(v, z) < 1/n$ for all n, and hence $v = z$.

Note that an upper bound in (1) is actually a maximal element.

Theorem 4(i) is due to Turinici [26], [27], [28] in a more general form. For far reaching generalization of Theorem 4(i) is obtained recently by Kang and Park [15]. The essential features of those generalizations imply well-known ordering principles of Altman [3], Brézis-Browder [7], Brϕndsted [8], and many others. Those principles unify a number of diverse results in nonlinear analysis. Note also that (i)$\Longrightarrow$(iii) is given by Turinici [26].

The condition (2) is implied by

(2)′ any nondecreasing sequence $\{x_n\}$ is *d-asymptotic* (that is, $\lim\inf_n d(x_n, x_{n+1})=0$).

For complete metric spaces, the following types of maximum principles have been widely used, e.g. the Banach contraction principle, the Bishop-Phelps lemma [4], and many of their extensions.

Theorem 5. *Let (X_0, d) be a metric space, $\phi: X_0 \longrightarrow \{-\infty\}\cup \boldsymbol{R}$ u.s.c., bounded above, and $k>0$. Define a partial order $\leq$ on $X=\{x\in X_0 \mid \phi(x)>-\infty\}$ by*

$$x\leq y \text{ iff } kd(x,y)\leqq \phi(y)-\phi(x).$$

Let $a\in X$, and suppose $S(a)$ is $\leq$-complete (that is, every nondecreasing Cauchy sequence converges).

Then (i)—(v) *of Theorem* 2 *hold.*

Proof. (i) We claim that any nondecreasing sequence $\{x_n\}$ is Cauchy. In fact, if $m\geqq n$, then $kd(x_n, x_m)\leqq \phi(x_m)-\phi(x_n)$ implies $\phi(x_m)\geqq \phi(x_n)$. Since $\phi(x_n)$ is nondecreasing and ϕ is bounded above, $\phi(x_n)\uparrow c$ for some $c\in \boldsymbol{R}$. This shows that $\{x_n\}$ is Cauchy and hence satisfies (2)′. Since $S(a)$ is $\leq$-complete, if $\{x_n\}$ is in $S(a)$, then $x_n \longrightarrow x$ for some $x\in S(a)$. In fact, $x\in X$, for $\phi(x)\geqq \lim\sup_n \phi(x_n)=c$ since ϕ is u.s.c. Now we claim that x is an upper bound of $\{x_n\}$. In fact

$$\begin{aligned} kd(x, x_n) &= \lim_{m\geq n} kd(x_m, x_n) \\ &\leqq \lim_{m\geq n} \sup \phi(x_m)-\phi(x_n) \\ &= a-\phi(x_n)\leqq \phi(x)-\phi(x_n). \end{aligned}$$

Therefore, by Theorem 3, we have the conclusion.

In Theorem 5, since ϕ is u.s.c., $S(a)$ is closed. Therefore, if (X_0, d) is complete, then clearl $S(a)$ is complete and hence $\leq$-complete. Moreover, if we choose $a\in X$ such that $\phi(a)\leqq \sup_X \phi - k$ then $S(a)=\{x\in X \mid kd(x,a)\leqq \phi(x)-\phi(a)\}\subset \{x\in X \mid \phi(x)>\phi(a),\ d(x,a)\leqq 1\}$. This gives mor accurate informations on whereabouts of locations of maximal points or fixed points.

Theorem 5(i) is due to Phelps [23] and extends the well-known Bishop-Phelps argument i [4]. Actually, Phelps proved (i) by using Zorn's lemma.

A dual form of Theorem 5 can be stated as follows:

Theorem 6. *Let (X_0, d) be a metric space, $\phi: X_0 \rightarrow \boldsymbol{R}\cup\{+\infty\}$ l.s.c., bounded below, ar $k>0$. Define a partial order $\leq$ on $X=\{x\in X_0 \mid \phi(x)<+\infty\}$ by*

$$x\leq y \text{ iff } kd(x,y)\leqq \phi(x)-\phi(y).$$

Let $a\in X$, and suppose $S(a)$ is $\leq$-complete. Then (i)—(v) *of Theorem* 2 *hold.*

Proof. Take $-\phi$ instead of ϕ and apply Theorem 5.

In Theorem 6, since ϕ is l.s.c., $S(a)$ is closed. Therefore, if (X_0, d) is complete, then clear $S(a)$ is complete and hence $\leq$-complete. Moreover, if we choose $a\in X$ such that $\phi(a)\leqq \inf_X\phi+$

then $S(a)=\{x\in X|\ kd(x,a)\leqq\phi(a)-\phi(x)\}\subset\{x\in X|\phi(x)<\phi(a),\ d(x,a)\leqq 1\}$.

Theorem 6 is given in Park [20], [21]. Actually, Theorem 6(i) is the celebrated variational principle of Ekeland [13], (ii) essentially due to Tuy [29], (v) to Kasahara [16], (iv) to Maschler-Peleg [18], and (iii) to Caristi-Kirk-Browder [10], which includes the Banach contraction principle. Applications of Theorem 6 are numerous in a vast field of mathematical sciences (see, e.g., [13], [20], [21]).

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