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P-Capacity and Hausdorff measure

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1. Introduction.

In paper[1], equivalence of p-Capacity and p-module is proved. Using it, we shall give a proof for the following result: If $1<p<n$, let a compact set $K\subset R^n$ have a finite Hausdorff measure for $h(r)=r^{n-p}$, then $Cap_p(A,K)=0$ for all open sets A containing K.
A condensor in the Euclidean space R^n is a pair $E=(A,K)$, where A is open in R^n and K is compact in A. For $p\geqq 1$, we define the p-Capacity of E as

$$Cap_pE=\inf_u\int |\nabla u|^p\, dm$$

where the infinimum is taken over all functions u in $C_0^\infty(A)$ such that $u(X)=1$ for all $x\in k$ and m is n-dimensional Lebesgue measure. Also, the Capacity of a condensor can be defined by module of path families. Now H^n will denote n-dimensional Hausdorff measure in R^n. Given a bounded condensor $E=(A,K)$, we let Γ_E be the family of all paths $\alpha:[a,b)\longrightarrow A$ such that $\alpha(a)\in K$ and $\alpha(t)\longrightarrow\partial A$ as $t\longrightarrow b$.
The p-dimensional module of Γ_E is

$$M_p(\Gamma_E)=\inf\{\int_{R^n} f^p dm : f\wedge\Gamma_E\}$$

where $f\wedge\Gamma_E$ means that f is a nonnegative m-measurable function satisfying the condition $\int_\beta f dH'\geqq 1$ for every $\beta\varepsilon\Gamma_E$. then $Cap_pE=M_p(\Gamma_E)$. by W. P. Ziemer [1]

2. Notation.

If $K\subset R^n$ and $r>0$, we let $B(K,r)$ be the set of all x in R^n such that $dist(x,K)<r$. In particular, $B(x,r)$ is the open ball with center at x and radius r. If K is compact, $E(K,r)$ will denote the condensor$(B(K,r), K)$.

Lemma 1. *If $Cap_p(A_0, K)=0$ for some bounded set A_0, then $Cap_p(A,K)=0$ for all open sets A containing K.*

Proof. Let $\{u_i\}$ be a sequence of $C_0^\infty(A_0)$, identically 1 on K. Suppose that $\|\nabla u_i\|_p\longrightarrow 0$ as $i\longrightarrow\infty$. Let ϕ be in $C_0^\infty(A)$ that equal 1 on K, then ϕu_i is in $C_0^\infty(A)$ that equal 1 on$K$, $\|\nabla(\phi u_i)\|_p\leqq\|\nabla\phi\cdot u_i\|_p+\|\phi\cdot\nabla u_i\|_p$. Using the Sobolev Inequality,

$$\|U_i\|_r\leqq cons\,\|\nabla U_i\|_p\,(r=\frac{np}{n-p})$$

Since $\|\nabla U_i\|_r\longrightarrow 0$, $\|U_i\|_r\longrightarrow 0$ and $u_i\longrightarrow 0$ a. e.

Since the u_i can be assummed to be bounded aboved by 1, $\| \nabla \phi \cdot u_i \|_p \longrightarrow 0$ *as* $i \longrightarrow \infty$ and since $\| \phi \cdot \nabla u_i \|_p \leqq cons \| \nabla u_i \|_p \longrightarrow 0$, $\| \nabla (\phi u_i) \|_p \longrightarrow 0$, Therefore $Cap_p(A, K) = 0$

Lemma 2. *If* $p > n$, *then* $Cap_p K = 0$ *only in the case* $K = \phi$.

Lemma 3. *If* $p > 1$ *and* K *is a compact set in* R^n_r *if* $Cap_p(A, K) > 0$ *for all open sets* A *containing* K, *then* $\lim_{r \to 0} Cap_p E(K, r) = \infty$.

Proof. Refer reference [3].

For any function $h: (0, 1) \longrightarrow (0, \infty)$, we let Λ_h denote the corresponding Hausdorff measure. *Set* $\Gamma(r) = \Gamma E(K, r)$.

Theorem. *Let* $1 < p < n$, *let* $h(r) = r^{n-p}$ *and let* K *be a compact set in* R^n *such that* $\Lambda_h(K) < \infty$, *then* $Cap_p(A, K) = 0$ *for all open sets* A *containing* K.

Proof. By Lemma 3, it suffices to show that $Cap_p E(K, r) = M(\Gamma(r))$ is bounded for small r. Let $0 < r < 1$, set $a = \Lambda_h(K)$ and choose a coutable covering of K with ball $B_i = B(x_i, r_i)$ with $x_i \in K$ and $r_i < r^2$ such that $\sum_i (r_i)^{p-n} \leqq a + 1$. Let Γ_i be the family of all path joining the boundary components of the spherical annulus $S(x, r) / s_i(x_i, r_i)$, then $\Gamma(r)$ is covered by $\cup \{ \Gamma_i \mid i \in N \}$. By [4, Theoreml 1], this implies

$$M(\Gamma(r)) \leqq \sum_i M(\Gamma_i) = W \sum_i \left(\frac{\alpha}{r_i^{-\alpha} - r^{-\alpha}} \right)^{p-1}, \qquad \alpha = \frac{n-p}{p-1}$$

where w is the (n-1)-area of the unit sphere.

$$\frac{\alpha}{r_i^{-\alpha} - r^{-\alpha}} = \frac{r^{\alpha} . \alpha}{\dfrac{r^{\alpha}}{r_i^{\alpha}} - 1} \leqq \frac{r^{\alpha} . \alpha}{\left(\dfrac{r}{r_i} \right)^{\alpha}} \leqq \alpha . (r_i)^{\alpha}.$$

Hence

$$M(\Gamma(r)) \leqq W \cdot \sum_i (\alpha (r_i)^{\alpha})^{p-1} = W \cdot \alpha^{p-1} \cdot \sum_i r_i^{n-p} \leqq W \cdot \alpha^{p-1} (a+1)$$

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