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Some Common Fixed Point Theorem For Jointly Densifying Mappings

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1. Introduction

Let A denote a bounded subset of a metric space (X, d). The infimum $d(A)$ of all $\varepsilon>0$ such that A admits a finite covering consiting of its subsets with diameters less than ε is called the measure of noncompactness of A [3].

In [1], Furi and Vignoli introduced the class of densifying mappings such as.

Definition 1. A continuous self mapping T of a metric space (X, d) is said to be densifying if, for every bounded subset A of X with $d(A)>0$, $d(T(A))<d(A)$.

For a densifying T, they proved

Theorem A. *Let T be a densifying mapping of a complete metric space (X, d) into itself and let F be a real valued lower semicontinuous function from $X\times X$ into $[0, \infty)$ such*

$$F(Tx, Ty)<F(x, y) \text{ for all } x, y\in X \text{ with } x\neq y.$$

If, for some $x\in X$, the sequence $\{x, Tx, T^2x, \cdots\}$ is bounded, then T has a unique fixed point in X.

Recently Khan, M. S. [2] has introduced the concept of jointly densifying mappings and proved a fixed point theorem for those mappings.

Defintion 2. Two continuous self mappings S and T of a metric space(X, d) are said to be jointly densifying if for every bounded subset A of X with $d(A)>0$, we have $d(ST(A))<d(A)$.

Theorem B. *Let S and T be jointly densifying mappings of a complete metric space(X, d) into itself and let F be a real valued lower semi-continuous function from $X\times X$ into $[0, \infty)$ such that*

$F(Sx, Ty)<F(x, y)$ and $F(Tx, Sy)<F(x, y)$ for all $x, y\in X$ with $x\neq y$. If, for some $x\in X$, the sequence $\{x_n\}$ defined by

$$Tx_{2n}=x_{2n+1},\quad Sx_{2n+2}=X_{2n+2}$$

for $n=0, 1, 2, \cdots$ is bounded, then either S or T has a fixed point.

The purpose of the present is to give a sufficient condition for a pair of jointly densifying mappings to have a unique common fixed point.

2. The common fixed point theorem

Theorem. *Let S and T be a pair of jointly densifying mappings of a complte metric space (X, d) into itself and let F be a real valued lower semi-continuous function from $X \times X$ into $[0, \infty)$ such that*

(1) $F(STx, TSy) < F(x, y)$ and $F(TSx, STy) < F(x, y)$ whenever $x \neq y$. If, for some $x \in X$, the joint sequence $\{x, Tx, STx, TSTx, STSTx, \cdots\}$ of iterates of x is bounded, then both S and T have fixed points.

If, in addition, F satisfies

(2) $F(Sx, Ty) < F(x, y)$ or $F(Tx, Sy) < F(x, y)$ whenever $x \neq y$, then S and T have a unique common fixed point in X.

Proof. Let $A = \{x, STx, STSTx, \cdots\}$, then A, the closure of A, is compact [2]. We know immediately that ST is stable on A. Define a real valued function ϕ on A by $\phi(x) = F(x, Tx)$, then, as in [2], ϕ has a minimum value at some point z in A. Now the point z is a fixed point of T, because, if $z \neq Tz$, we have

$$(ST_z) = F(ST_z, TST_z) < F(z, T_z) = \phi(z),$$

which contradicts the minimality of $\phi(z)$.

On the other hand, let $B = \{Tx, TSTx, TSTSTx, \cdots\}, \cdots\}$, then $B = T(A)$ so that B is contained in the compact set $T(A)$ which is also closed since X is a metric space. But then B is compact. By a similar method we can pick out a fixed point of S in X. Hence we conclude that both S and T have fixed points.

Now, assume (2), say $F(Sx, Ty) < F(x, y)$ if $x = y$. Let x_o and y_o be fixed points of S and T respectively. If $x_o \neq y_o$, then we have $F(x_o, y_o) = F(Sx_o, Ty_o) < F(x_o, y_o)$, a contradiction. Therefore it must be that $x_o = y_o$.

Suppose that x_1 is another fixed point of T(or S) such that $x_1 \neq x_0$. then $F(x_0, x_1) = F(Sx_0, Tx_1) < F(x_0, x_1)$ ($F(x_1, x_0)$ $F(Sx_1, Tx_0) < F(x_1, x_0)$ resp.). This contradiction assures that x_o is the unique common fixed point of S and T. and

Remark. by letting $S = 1x$, the identity mapping of X and T be a densifying mapping, we have, as a corollary, Theorem A of Furi and Vignoli.

References

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