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On Some Properties of Expansive Homeomorphisms on σ-Compact Metric Spaces

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1. Introduction

Let (X, d) be a metric space. Then a homeomorphism f of X onto itself is called an expansive homeomorphism if there exists a positive number c such that for each pair (x, y) of distinct points of X, there exists an integer n for which $d(f^n(x), f^n(y)) > c$. The number c is called an expansive constant for f.

Since 1950, several papers have been written which are concerned, with this property possessed by an expansive homeomorphism on a metric space, in particular, compact metric space. W. R. Utz[4], B. F. Bryant[1] and W. Reddy[2] proved that if a metric space X is compact and dense-in-itself, and $f: X \to X$ is an expansive homeomorphism, then there exist at most finite fixed points and countably many periodic points under f, and the set of points having converging semi-orbits under f is at most countable.

In this paper, more generally, we will show that if a metric space X is *σ-compact* and dense-in-itself, and $f: X \to X$ is an expansive homeomorphism, then there exist at most countably many fixed points and periodic points under f, and the set of points having converging semi-orbits under f is at most countable.

From now, unless otherwise qualified, (X, d) shall denote a metric space with metric d that is dense-in-itself. Here, a space is said to be dense-in-itself if every point of the space is a limit point of the space.

2. Main results

Theorem 1. *Let (X, d) be a σ-compact metric space and $f: X \to X$ be an expansive homeomorphism. Then the set of fixed points and the set of periodic points under f are at most countable, respectively.*

Proof. Let $X = \bigcup_{n=1}^{\infty} X_n$ (X_n: compact spaces) and $F(f) = \{p \in X \mid f(p) = p\}$. Then $F(f) = \bigcup_{n=1}^{\infty} (X_n \cap F(f))$. Assume that $F(f)$ is not countable. Then there exists a positive integer n_o such that $Xn_o \cap F(f)$ is infinite. Since Xn_o is compact, Xn_o is closed and so $Xn_o \cap F(f)$ is compact. Let $\langle Xn \rangle$ be an infinite sequence in $Xn_o \cap F(f)$ such that $i \neq j$ implies $x_i \neq x_j$. Then there exists a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $\langle x_{n_k} \rangle$ converges to some $x \in$

$Xn_0 \cap F(f)$. For the expansive constant c of f, choose $x_{n_k} \in Xn_0 \cap F(f)$ such that $d(x_{n_k}, x) < c$. Then $d(x_{n_k}, x) = d(f^m(x_{n_k}), f^m(x))$ for all integer m. This is a contradiction and so $F(f)$ is at most countable.

Let $P(f)$ be the set of all periodic points of f and $F(f^n) = \{p \in X \mid f^n(p) = p\}$. Then obviously $P(f) = \bigcup_{n=1}^{\infty} F(f^n)$. Suppose that $P(f)$ is uncountable. Then $F(f^n)$ is uncountable for some integer n. Since $F(f^n) = F(f^n) \cap X = \bigcup_{m=1}^{\infty}(X_m \cap F(f^n))$, $X_m \cap F(f^n)$ is uncountable for some m. Now, we choose a sequence $\langle x_k \rangle$ and x in $X_m \cap F(f^n)$ such that $\langle x_k \rangle$ converges to x. Then

$$d(x_k, x) \to 0 \ \text{as} \ k \to \infty,$$
$$d(f(x_k), f(x)) \to 0 \ \text{as} \ k \to \infty,$$
$$d(f^2(x_k), f^2(x)) \to 0 \ \text{as} \ k \to \infty,$$
$$\vdots$$
$$d(f^{n-1}(x_k), f^{n-1}(x)) \to 0 \ \text{as} \ k \to \infty.$$

For the expansive constant c of f, there exist positive integers $N_0, N_1, \cdots\cdots, N_{n-2}$ and N_{n-1} such that

$$d(x_k, x) < c \ \text{for} \ k \geq N_0,$$
$$d(f(x_k), f(x)) < c \ \text{for} \ k \geq N_1,$$
$$d(f^2(x_k), f^2(x)) < c \ \text{for} \ k \geq N_2,$$
$$\vdots$$
$$d(f^{n-1}(x_k), f^{n-1}(x)) < c \ \text{for} \ k \geq N_{n-1}.$$

Let $N = max\{N_0, N_1, \cdots\cdots, N_{n-1}\}$. Then $d(f^j(x_N), f^j(x)) < c$ for all integers j. This is a contradiction and so $P(f)$ is at most countable.

If f is a homeomorphism of a metric space (X, d) onto itself and $x \in X$, then $O(x) = \{f^n(x) \mid n \in Z\}$ is called the orbit of x under f, $O_-(x) = \{f^n(x) \mid n \in Z_-\}$ is called the negative semi-orbit of x under f, and $O_+(x) = \{f^n(x) \mid n \in Z_+ \cup \{o\}\}$ is called the positive semi-orbit of x under f, where Z, Z_- and Z_+ denote the set of integers, negative integers and positive integers, respectively. The point y is said to be an α-limit point (w-limit point) of the orbit $O(x)$ if $y = \lim_{j \to \infty} f^{n_j}(x)$ for some strictly decreasing (increasing) sequence of integers n_j. We will denote the set of all α-*limit*(*w-limit*) points of the orbit $O(x)$ by $\alpha(x)(w(x))$. If for some point x, sequences $\langle f^n(x) \rangle$ and $\langle f^{-n}(x) \rangle$ converge, we say x has converging semi-orbits under f.

Theorem 2. *Let f be an expansive homeomorphism of a σ-compact metric space (X, d) onto itself. Then the set of points having converging semi-orbits under f is at most countable.*

Proof. Let c be an expansive constant for f. Since f is an expansive homeomorphism, f has at most countably many fixed points, say

$$q_1, q_2, q_3, \cdots\cdots\cdots\cdots\cdots\cdots.$$

Let A be the set of points having converging semi-orbits under f and suppose that A

is uncountable. If $x \in A$, then both $\lim_{n\to\infty} f^{-n}(x)$ and $\lim_{n\to\infty} f^{n}(x)$ are fixed points. Let $A_{ij} = \{x \in A \mid \lim_{n\to\infty} f^{-n}(x) = q_i$ and $\lim_{n\to\infty} f^{n}(x) = q_j\}$, where $1 \leq i, j$. Then $A = \bigcup_{1 \leq i,j} A_{ij}$ and one of these (call it $B = A_{i_0 j_0}$) is uncountable. For each positive integer N, define $B(N) = \{x \in B \mid n \geq N$ implies $d(f^{n}(x), q_{j_0}) < \frac{c}{2}$ and $d(f^{-n}(x), q_{i_0}) < \frac{c}{2}\}$. Then $B = \bigcup_{N=1}^{\infty} B(N)$. Since B is mncountable, there exists a positive integer N_0 such that $B(N_0)$ is uncountable. Let $X = \bigcup_{m=1}^{\infty} X_m$ (X_m; compact spaces). Then $B(N_0) = \bigcup_{m=1}^{\infty} (B(N_0) \cap X_m)$. Since $B(N_0)$ is uncountable, there exists a compact space X_m such that $B(N_0) \cap X_m$ is uncountable(infinite). Since X_m is compact, f^{n} is uniformly continuous on X_m for each integer n with $|n| \leq N_0$. Thus for each integer n with $|n| \leq N_0$, there exists $\delta_n > 0$ such that $d(x, y) < \delta_n$ implies $d(f^{n}(x), f^{n}(y)) < c$. Let $\delta = min\{\delta_n \mid |n| \leq N_0\}$. Since $B(N_0) \cap X_m \subset X_m \subset \bigcup_{x \in X_m} S(x, \frac{\delta}{2})$ and X_m is compact, there exist $x_1, x_2, \cdots\cdots, x_{m_0}$ *in* X_m such that $B(N_0) \cap X_m \subset X_m \subset \bigcup_{k=1}^{m_0} S(x_k, \frac{\delta}{2})$, where $S(x_k, \frac{\delta}{2}) = \{y \in X \mid d(x_k, y) < \frac{\delta}{2}\}$. Also, there exists k_0 with $1 \leq K_0 \leq m_0$ such that $S(x_{k_0}, \frac{\delta}{2})$ has uncountable elements of $B(N_0) \cap X_m$. Take $y, z \in S(x_{k_0}, \frac{\delta}{2}) \cap B(N_0) \cap X_m$. Then $d(y, z) \leqq d(y, x_{k_0}) + d(x_{k_0}, z) < \frac{\delta}{2} + \frac{\delta}{2} = \delta$. Thus $|n| \leqq N_0$ implies $d(f^{n}(y), f^{n}(z)) < c$. Since $y, z \in B(N_0)$, $n \geqq N_0$ implies $d(f^{n}(y), f^{n}(z)) \leqq d(f^{n}(y), q_{j_0}) + d(q_{j_0}, f^{n}(z)) < \frac{c}{2} + \frac{c}{2} = c$ and $d(f^{-n}(y), f^{-n}(z)) \leqq d(f^{-n}(y), q_{i_0}) + d(q_{i_0}, f^{-n}(z)) < \frac{c}{2} + \frac{c}{2} = c$. Thus we conclude that $d(f^{n}(y), f^{n}(z)) < c$ for all integers n. This is a contradiction to the choice of c. Therefore, A is countable.

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