韓國數學教育學會誌〈數學教育〉
1987. 12, Vol. XXVI, No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1987, Vol. XXVI, No. 1

A Note on L^p-Spaces

by Jung-Rye Lee

Ph. D. Students of Seoul National University, Seoul, Korea

1. Introduction

We derive characterizations of the measure space (X, Λ, μ) such that $L^p(\mu)$ and $L^q(\mu)$, for $p, q \in (0, \infty)$ contains the same functions. And we take some examples with $L^p(\mu) = L^q(\mu)$.

In this paper, (X, Λ, μ) will be a positive measure space and, for each $p \in (0, \infty]$, $L^p(\mu)$ will denote the space of all Λ-measurable real functions f on X such that $\|f\|_p < \infty$, where $\|f\|_p = (\int_X |f|^p d\mu)^{1/p}$ for $p \in (0, \infty)$, and $\|f\|_\infty = ess \sup_X |f(x)|$ and we identify two functions which differ on a set of measure zero.

If we give the metric dp, $dp(f, g) = \|f - g\|_p$, for $p \in (1, \infty]$, then $L^p(\mu)$ for $p \in (1, \infty]$, becomes a complete metric space.

2. Results

Lemma 2. 1. *Let $p, q \in [1, \infty]$, the set inclusion $L^p(\mu) \subset L^q(\mu)$ implies that the inclusion map $i: L^p(\mu) \to L^q(\mu)$ is continuous.*

Proof. If $f_n \to f$ in $L^p(\mu)$, then $\{f_n\}$ has a subsequence which converges pointwise almost everywhere to f, by Theorem 3. 12, [1]. And, from the closed Graph Theorem, Theorem 2, 15, [2], the inclusion map is continuous.

Theorem 2. 2. *Let (X, Λ, μ) be a positive measure space with $\inf_{E \in \Lambda_0} \mu(E) > 0$ and $\sup_{E \in \Lambda_\infty} \mu(E) < \infty$, where $\Lambda_0 = \{E \in \Lambda \mid \mu(E) > 0\}$ and $\Lambda_\infty = \{E \in \Lambda \mid \mu(E) < \infty\}$, Then $L^p(\mu) = L^q(\mu)$ for any $p, q \in (0, \infty)$.*

Proof. By the following Propositions, this is trivial.

Proposition 2. 3. *If $\mu(X) < \infty$, then the followings are equivalent.*

(1) *$L^p(\mu) \subset L^q(\mu)$ for some $p, q \in (0, \infty]$ with $p < q$,*

(2) *$\inf_{E \in \Lambda_0} \mu(E) > 0$,*

(3) *$L^p(\mu) \subset L^q(\mu)$ for all $p, q \in (0, \infty]$ with $p < q$,*

(4) *Any collection of disjoint measurable sets of positive measure is finite.*

Proof. (1) ⇒ (2). Assume $L^p(\mu) \subset L^q(\mu)$ and $f \in L^{pt}(\mu)$ for every $t \in (0, \infty)$, then $|f|^t \in L^p \subset L^q$, i. e., $\int |f|^{qt} d\mu < \infty$. Thus, we have $L^{pt}(\mu) \subset L^{qt}(\mu)$ for every $t \in (0, \infty)$ and we can assume $p \geqq 1$. By Lemma 2.1., there is a constant $k>0$ such that $\|f\|_q \leqq k\|f\|_p$ for any $f \in L^p(\mu)$. By taking $f=X_E$ for any $E \in A_0$, we have $(\mu(E))^{1/q} \leqq k(\mu(E))^{1/p}$ and hence $k^{-1} \leqq \mu(E)^{1/p-1/q}$, i.e., $k^{pq/p-q} \leqq \mu(E)$. It follows that $inf_{E \in A_0} \mu(E) > 0$.

(2) ⇨ (3). Let $f \in L^p(\mu)$ and $E_n = \{x: |f(x)| > n\}$ $n=1,2,\cdots\cdots$. Since $\infty > \int |f|^p d\mu \geqq \int_{En} |f|^p d\mu \geqq n^p \mu(E_n)$, we have $\mu(E_n) \to 0$ as $n \to \infty$. By condition (2), there is an index n_0 such that $\mu(E_n)=0$ for $n \geqq n_0$, i.e. $|f| \leqq n_0$ $\mu-a.e.$ This means $f \in L^\infty(\mu)$ and since $\mu(X) < \infty$, $f \in L^q(\mu)$ for every $q \in [p, \infty]$.

(3) ⇨ (4). Suppose $\{E_n\}$ is a sequence of disjoint measurable sets such that $\mu(E_n) \neq 0$ for infinitely many n. Since $\mu(\bigcup_1^\infty E_n) = \sum_1^\infty (E_n) \leqq \mu(X) < \infty$, we must have $\mu(E_n) \to 0$ as $n \to \infty$. Let $\{E_{n_k}\}$ be a subsequence such that $0 < \mu(E_{n_k}) < 1/k^3$ and we take $h: X \to R$ to be $h(x)=k$ for $x \in E_{n_k}$ and $h(x)=0$ for $x \notin \bigcup_k E_{n_k}$. Then, since $\mu(E_{n_k}) > 0$ for every k, $f \notin L^\infty(\mu)$ and $\int_X |h| d\mu = \sum_{k=1}^\infty k\mu(E_{n_k}) < \sum_{k=1}^\infty 1/k^2 < \infty$, i.e., $h \in L^1(\mu)$.

This contradicts to our hypothesis.

(4) ⇨ (1). The condition of (4) implies $f \in L^\infty(\mu)$ for every measurable function f. Since $\mu(X) < \infty$, it follows that $f \in L^p(\mu)$ for any $p > 0$.

Proposition 2.4. *If $\mu(X) = \infty$ then followings are equivalent.*

(1) *$L^p(\mu) \subset L^q(\mu)$ for some $p, q \in (0, \infty]$ with $p < q$,*

(2) *$inf_{E \in A_0} \mu(E) > 0$,*

(3) *$L^p(\mu) \subset L^q(\mu)$ for all $p, q \in (0, \infty]$ with $p < q$.*

Proof. (1) ⇨ (2). As in proof of Proposition 2.3.

(2) ⇨ (3). For any $f \in L^p(\mu)$, set $An = \{x: n < |f(x)| \leqq n+1\}$ $n=0,1,2,\cdots\cdots$, so that $\{A_n\}$ is a disjoint collection of measurable sets. If $\mu(A_n) \neq 0$ for infinitely many n, then $\mu(A_n) \geqq c > 0$ for some c, by hypothesis and hence $\infty = \sum_{n=1}^\infty n^p \mu(A_n) \leqq \sum_{n=0}^\infty \int_{A_n} |f|^p d\mu = \|f\|_p^p < \infty$.

Thus there is an index n_0 such that $\mu(E^n)=0$ for $n \geqq n_0$. Since on A_0, $|f|^q \leqq |f|^p$ and for $1 \leqq n \leqq n_0$,

$$(n+1)^q = (n+1)^{q-p}(n+1)^p \leqq (n_0+1)^{q-p}(2n)^p = 2^p(n_0+1)^{q-p} n^p,$$

$$\int |f|^q d\mu = \int_{A_0} |f|^q d\mu + \sum_{n=1}^{n_0} \int_{A_n} |f|^q d\mu$$

$$\leqq \int_{A_0} |f|^p d\mu + \sum_{n=}^{n_0} (n+1)^q \mu(A_n)$$

$$\leqq \int_{A_0} |f|^p d\mu + 2^p(n_0+1)^{q-p} \sum_{n=1}^{n_0} n^p \mu(A_n)$$

$$\leqq (1+2^p(n_0+1)^{q-p})(\int |f|^p d\mu) < \infty.$$

Hence $f \in L^q(\mu)$.

(3) ⇨ (1). This is trivial.

Proposition 2.5. *The followings are equivalent.*

(1) *$L^p(\mu) \supset L^q(\mu)$ for some $p, q \in (0, \infty)$ with $p < q$,*

(2) $sup_{E\epsilon\Lambda_\infty}\mu(E)<\infty$,

(3) $L^p(\mu)\supset L^q(\mu)$ *for all* $p, q\epsilon(0,\infty)$ *with* $p<q$.

Proof. (1) ⇒ (2). As in the proof of Proposition 2.3., we can assume $p\geqq 1$ and by Lemma 2.1., there is a constant $k>0$ such that $\|f\|_p\leqq k\|f\|_q$ for every $f\epsilon L^q(\mu)$. It follows that $\mu(E)\leqq k^{pq/p-q}$ for every $E\epsilon\Lambda_\infty$, which means $sup_{E\epsilon\Lambda_\infty}\mu(E)<\infty$.

(2) ⇒ (3). Let $f\epsilon L^q(\mu)$ and $E_n=\{1/(n+1)\leqq|f|<1/n\}$ $n=1,2,\cdots\cdots$. Then $\int_X|f|^q d\mu\geqq \int_{E_n}|f|^q d\mu\geqq(1/n+1)^q\mu(E_n)$, *i. e.*, $\mu(E_n)\leqq(n+1)^q\int|f|^q d\mu<\infty$ and hence $\sum_{n=1}^{\infty}\mu(E_n)\leqq sup_{E\epsilon\Lambda_\infty}\mu(E)<\infty$, by condition of (2).

Thus
$$\int_X|f|^p d\mu=\int_{\{|f|\geq 1\}}|f|^p d\mu+\sum_{n=1}^{\infty}\int_{E_n}|f|^p d\mu$$
$$\leqq\int_X|f|^q d\mu+\sum_{n=1}^{\infty}1/n^p\mu(E_n)$$
$$\leqq\int_X|f|^q d\mu+\sum_{n=1}^{\infty}\mu(E_n)<\infty.$$

(3) ⇒ (1). This is trivial.

3. Examples

We take some examples which give a meaning of our theorem.

If we take measure spaces as follows, then clearly $L^p(\mu)=L^q(\mu)$ for $p, q\epsilon(0,\infty)$.

(1) For any $E\epsilon\Lambda$, $\mu(E)=0$,

(2) A counting measure μ on a finite set,

(3) A unit mass measure concentrating at $x_0\epsilon X$,

(4) Let X be a uncountable set and Λ is a collection of $E\subset X$ such that E or E^c is at most countable. Define μ,

$$\mu(E)=\begin{cases}0 & \text{if } E \text{ is at most countable}\\ 1 & \text{if } E^c \text{ is at most countable.}\end{cases}$$

But all of these measures are finite measure.

Finally, we take measure space (X,Λ,μ) such that $\mu(X)=\infty$ and $L^p(\mu)=L^q(\mu)$ for $p, q\epsilon(0,\infty)$.

(5) Let $X=[0,N]$, $I_n=(n-1,n]$, $n=1,2,\cdots\cdots,N$ and $\Gamma=\{\phi, I_1, I_2,\cdots\cdots,I_N\}$ be a collection of subsets of X. By Theorem 1.10, [1], there exists the smallest *σ-algebra* Λ containing Γ. For any $E\epsilon\Lambda$, $E\subset[0,N]$ such that $E=\bigcup_{k\epsilon F}J_k$, $J_k=I_k$ or $\{0\}$ where F is a finite set. Now define μ, $\mu(E)=\begin{cases}m(E) & E\neq X\\ \infty & E=X\end{cases}$ where m is Lebesgue measure, then $inf_{E\epsilon\Lambda_0}\mu(E)=1$ and $sup_{E\epsilon\Lambda_\infty}\mu(E)=N$.

Hence, by Theorem 2.2., $L^p(\mu)=L^q(\mu)$ *for* $p, q\epsilon(0,\infty)$.

References

1. W. Rudin, *Real and complex Analysis*, 2nd ed., McGraw-Hill, 1974.
2. ———, *Functional Analysis*, McGraw-Hill, 1973.