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Common Fixed Point Theorems of Commuting Mappinggs

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Abstract In this paper, we give several fixed point theorems in a complete metric space for two multi-valued mappings commuting with two single-valued mappings. In fact, our main theorems show the existence of solutions of functional equations $f(x)=g(x)\in Sx\cap Tx$ and $x=f(x)=g(x)\in Sx\cap Tx$ under certain conditions. We also answer an open question proposed by Rhoades-Singh-Kulsherestha.

Throughout this paper, let (X, d) be a complete metric space. We shall follow the following notations:

$CL(X)=\{A;\ A$ is a nonempty closed subset of $X\}$,

$CB(X)=\{A;\ A$ is a nonempty closed and bounded subset of $X\}$,

$C(X)=\{A;\ A$ is a nonempty compact subset of $X\}$.

For each $A,\ B\in CL(X)$ and $\varepsilon>0$,

$$N(\varepsilon, A)=\{x\in X;\ d(x, a)<\varepsilon \text{ for some } a\in A\},$$

$$E_{A,B}=\{\varepsilon>0;\ A\subset N(\varepsilon, B) \text{ and } B\subset N(\varepsilon, A)\},$$

and

$$H(A, B)=\begin{cases} \inf E_{A,B}, & \text{if } E_{A,B}\neq\phi, \\ +\infty, & \text{if } E_{A,B}=\phi. \end{cases}$$

Then H is called the generalized Hausdorff distance function for $CL(X)$ induced by a metric d and H defined $CB(X)$ is said to be the Hausdorff metric induced by d. $D(x, A)$ will denote the ordinary distance between $x\in X$ and a nonempty subset A of X. Let $\mathbb{R}^+$ and $\mathbb{I}^+$ denote the sets of nonnegative real numbers and positive integers, respectively, and G the family of functions ϕ from $(\mathbb{R}^+)^5$ into $\mathbb{R}^+$ satisfying the following conditions:

(1) ϕ is nondecreasing and upper semicontinuous in each coordinate variable, and

(2) for each $t>0$, $\psi(t)=\max\{\phi(t, 0, 0, t, t), \phi(t, t, t, 2t, 0), \phi(0, t, 0, 0, t)\}<t$, *where* $\psi: \mathbb{R}^+\to\mathbb{R}^+$ is a nondecreasing and upper semicontinuous function from the right.

Before stating and proving our main theorems, we give the following lemmas:

Lemma 1. [5] *Let* $\psi: \mathbb{R}^+\to\mathbb{R}^+$ *be a nondecreasing* and upper semicontinuous function from the right. Then $\lim_{n\to\infty}\psi^n(t)=0$ if and only if $\psi(t)<t$ for all $t>0$, where ψ^n is the n-th iteration of ψ.

Lemma 2. [11] *Let A, $B \in CB(X)$. Then for each $\varepsilon > 0$ and $a \in A$, there exists a point $b \in B$ such that $d(a, b)$ $H(A, B) + \varepsilon$. Furthermore, if A, $B \in C(X)$, then we can select a point $b \in B$ such that $d(a, b) \leqq H(A, B)$.*

Now, we are ready to give our main theorems:

Theorem 3. *Let S and T be multi-valued mappings from X into $C(X)$. If there exist continuous single-valued mappings f and g from X into itself such that*

(3) $T(X) \subseteq g(X)$ and $S(X) \subseteq f(X)$,

(4) $fTx = Tfx$ and $gSx = Sgx$ for all $x \in X$,

(5) $H(Tx, Sy) \leqq \phi(d(fx, gy), D(fx, Tx), D(gy, Sy), D(fx, Sy), D(gy, Tx))$, for all x, $y \in X$ and some $\phi \in \mathcal{G}$, then f, g, S and T have a coincidence point in X. Further, if z is a coincidence point of f, g, S and T and fz is a fixed point of f (or gz is a fixed point of g), then fz is a common fixed point of f, g, S and T.

Proof. Pick any fixed point $x_0 \in X$. Since $S(X) \subseteq f(X)$, we can find a point $x_1 \in X$ such that $fx_1 \in Sx_0$. Noting that $T(X)$ is also a subset of $g(X)$, by Lemma 2, for a suitable point $x_2 \in X$, we can choose a point $gx_2 \in Tx_1$ such that $d(gx_2, fx_1) \leqq H(Tx_1, Sx_0)$.
In general, we can choose a sequence $\{x_n\}$ in X such that $fx_{2n-1} \in Sx_{2n-2}$, $gx_{2n} \in Tx_{2n-1}$, $fx_{2n+1} \in Sx_{2n}$ and $d(fx_{2n-1}, gx_{2n}) \leqq H(Sx_{2n-2}, Tx_{2n-1})$, $d(gx_{2n}, fx_{2n+1}) \leqq H(Tx_{2n-1}, Sx_{2n})$.
Then by (5), we have

$$\begin{aligned} & d(gx_{2n}, fx_{2n+1}) \\ & \leqq H(Tx_{2n-1}, Sx_{2n}) \\ & \leqq \phi(d(fx_{2n-1}, gx_{2n}), D(fx_{2n-1}, Tx_{2n-1}), D(gx_{2n}, Sx_{2n}), D(fx_{2n-1}, Sx_{2n}, D(gx_{2n}, Tx_{2n-1})) \\ & \leqq \phi(d(fx_{2n-1}, gx_{2n}), d(fx_{2n-1}, gx_{2n}), d(gx_{2n}, fx_{2n+1}), d(fx_{2n-1}, fx_{2n+1}), 0) \\ & \leqq \phi(d(fx_{2n-1}, gx_{2n}), d(fx_{2n-1}, gx_{2n}), d(gx_{2n}, fx_{2n+1}), d(fx_{2n-1}, gx_{2n}) + d(gx_{2n}, fx_{2n+1}), \\ & \quad 0). \end{aligned}$$

If $d(gx_{2n}, fx_{2n+1}) > d(fx_{2n-1}, gx_{2n})$,
then $d(gx_{2n}, fx_{2n+1})$

$$\begin{aligned} & \leqq \phi(d(fx_{2n+1}, gx_{2n}), d(fx_{2n+1}, gx_{2n}), d(gx_{2n}, fx_{2n+1}), 2d(fx_{2n+1}, gx_{2n}), 0) \\ & \leqq \psi(d(gx_{2n}, fx_{2n+1})) \\ & < d(gx_{2n}, fx_{2n+1}), \end{aligned}$$

which is a contradiction.
Thus, $d(gx_{2n}, fx_{2n+1})$

$$\begin{aligned} & \leqq \phi(d(fx_{2n-1}, gx_{2n}), d(fx_{2n-1}, gx_{2n}), d(fx_{2n-1}, gx_{2n}), 2d(fx_{2n-1}, gx_{2n}), 0) \\ & \leqq \psi(d(gx_{2n}, fx_{2n-1}). \end{aligned} \tag{6}$$

Similarly, we have

$$\begin{aligned} & d(fx_{2n+1}, gx_{2n+2}) \\ & \leqq \psi(d(gx_{2n}, fx_{2n+1})). \end{aligned} \tag{7}$$

If follows from (6) and (7) that

$$\begin{aligned} & d(fx_{2n+1}, gx_{2n+2}) \\ & \leqq \psi(d(gx_{n}, fx_{2n+1})) \\ & \leqq \psi^2(d(fx_{2n-1}, gx_{2n})) \leqq \cdots \\ & \leqq \psi^{2n-1}(d(fx_3, gx_2)) \end{aligned}$$

$$\leq \psi^{2n}(d(fx_1, gx_2)). \tag{8}$$

By Lemma 1 and (8), we have

$$d(fx_{2n+1}, gx_{2n+2})=0. \tag{9}$$

Similarly, we have $d(gx_{2n}, fx_{2n+1})=0$ (10)

For positive even integers m, $n(m>n)$, $d(gx_m, gx_n)$

$\leq d(gx_m, fx_{m-1})+d(fx_{m-1}, gx_{m-2}))+\cdots+d(fx_{n+1}, gx_n)$, which shows that $\{gx_{2n}\}$ is a Cauchy sequence in X, where $n\in I^+$. Similarly, $\{fx_{2n+1}\}$ is also a Cauchy sequence in X. Thus, since (X, d) is a complete metric space, $\{fx_{2n+1}\}$ and $\{gx_{2n}\}$ converge to some points in X and, by (9) and (10), their limits are equal.

Let $\lim_{n\to\infty} fx_{2n+1}=\lim_{n\to\infty} gx_{2n}=z$.

Since f and g are continuous,

$$fgx_{2n} \longrightarrow fz,$$
$$ffx_{2n-1} \longrightarrow fz,$$
$$gfx_{2n+1} \longrightarrow gz,$$

and $ggx_{2n} \longrightarrow gz$, as $n \longrightarrow \infty$.

By (4) and (5), we have

$$\begin{aligned}
&d(fgx_{2n}, gfx_{2n+1})\\
&\leq H(fTx_{2n-1}, gSx_{2n})\\
&=H(Tfx_{2n-1}, Sgx_{2n})\\
&\leq \phi(d(ffx_{2n-1}, ggx_{2n}), D(ffx_{2n-1}, Tfx_{2n-1}), D(ggx_{2n}, Sgx_{2n}), D(ffx_{2n-1}, Sgx_{2n}),\\
&\quad D(ggx_{2n}, Tfx_{2n-1}))\\
&=\phi(d(ffx_{2n-1}, ggx_{2n}), D(ffx_{2n-1}, fTx_{2n-1}), D(ggx_{2n}, gSx_{2n}), D(ffx_{2n-1}, gSx_{2n}),\\
&\quad D(ggx_{2n}, fTx_{2n-1}))\\
&\leq \phi(d(ffx_{2n-1}, ggx_{2n}), d(ffx_{2n-1}, fgx_{2n}), d(ggx_{2n}, gfx_{2n+1}), d(ffx_{2n-1}, gfx_{2n+1}),\\
&\quad d(ggx_{2n}, fgx_{2n})), \text{ which implies that if } n \longrightarrow \infty,
\end{aligned}$$

$$\begin{aligned}
d(fz, gz) &\leq \phi(d(fz, gz), 0, 0, d(fz, gz), d(fz, gz))\\
&\leq \psi(d(fz, gz))\\
&< d(fz, gz), \text{ which is impossible.}
\end{aligned}$$

Thus, we have $fz=gz$.

Also, again by (4) and (5),

$$\begin{aligned}
D(gz, Tz) &\leq d(gz, gfx_{2n+1})+D(gfx_{2n+1}, Tz)\\
&\leq d(gz, gfx_{2n+1})+H(Tz, gSx_{2n})\\
&=d(gz, gfx_{2n+1})+H(Tz, Sgx_{2n})\\
&\leq d(gz, gfx_{2n+1})+\phi(d(fz, ggx_{2n}), D(fz, Tz), D(ggx_{2n}, Sgx_{2n}),\\
&\quad D(fz, Sgx_{2n}), D(ggx_{2n}, Tz))\\
&\leq d(gz, gfx_{2n+1})+\phi(d(gz, ggx_{2n}), D(gz, Tz), d(ggx_{2n}, gfx_{2n+1}),\\
&\quad d(gz, gfx_{2n+1}), D(ggx_{2n}, Tz)).
\end{aligned}$$

Letting $n \longrightarrow\infty$, this inequality obtains

$$\begin{aligned}
D(gz, Tz) &\leq \phi(0, D(gz, Tz), 0, 0, D(gz, Tz))\\
&\leq \psi(D(gz, Tz))\\
&< D(gz, Tz), \text{ which is impossible.}
\end{aligned}$$

Thus we have $gz \in Tz$. Similarly, we have $fz \in Sz$, that is, z is a coincidence point of f, g, s and T.

Next, since $f^2z = ffz \in fTz = Tfz$ and $fz = gz$, by (5), $d(f^2z, fz)$

$$\begin{aligned} &\leqq H(Tfz, Sz) \\ &= \phi(d(f^2z, gz), D(f^2z, Tfz), D(gz, Sz), D(f^2z, Sz), D(gz, Tfz)) \\ &\leqq \phi(d(f^2z, fz), 0, 0, d(f^2z, fz), d(fz, f^2z)) \\ &\leqq \psi(d(f^2z, fz)) \\ &< d(f^2z, fz), \text{ which is impossible.} \end{aligned}$$

Thus, we have $fz = f^2z$.

Therefore, $gz = fz$ *is* a common fixed point of f, g, S and T.

Remark 1. *Theorem 3 is a solution of an open problem proposed by Rhoades-Singh-Kulshrestha in [12]. As immediate consequences of Theorem 3, we have the following:*

Corollary 4. *Let S and T be multlvalued mappings from X into $C(X)$. If there exist continuous single-valued mappings f and g from X into itself such that*

(11) $S(X) \subseteqq f^s(X)$ and $T(X) \subseteqq g^t(X)$ for some fixed positive integers s and t,

(12) $fTx = Tfx$ and $gSx = Sgx$ for all $x \in X$,

(13) $H(T^px, S^qy) \leqq \phi(d(f^sx, g^ty), D(f^sx, T^px), D(g^ty, S^qy), D(f^sx, S^qy), D(g^ty, T^px))$ for all $x, y \in X$,

Where p and q are some positive integers, then the conclusion of Theorem 3 is valid.

Corollary 5. *Let f, g, S and T be as in Theorem 3. Suppore that (3), (4) and (5) of of Theorem 3 hold. If there exists a fundtion $\phi \in \mathcal{J}$ satisfying (1) and (14):*

(14) $\psi(t) = \phi(t, t, t, at, bt) < t$ for all $t > 0$, where $a + b = 2$, then the concluston of Theorem 3 holds.

Remark 2. *Theorem 3 and Corollary 4 extend some main results of [1], [3], [4], [7]-[9], [13] and [14] to multivalued mappings and the main theorem of [10] is a sepecial case of Corollary 5.*

Theorem 6. *Let S and T be multivalued mappings from X into $C(X)$. If there exist continuous mappings f and g from X into itself such that (4) and (5) of Theorem 3 hold and*

(15) $T(f^{t-1}(X)) \subseteqq g^s(X)$ and $S(g^{s-1}(X)) \subseteqq f^t(X)$ for some fixed positive integers s and t, then the bonclusion of Theorem 3 is valid.

Proof. Pich any point $x_0 \in g^{s-1}(X)$.

By the hypothesis, there exists a sequence $\{x_n\}$ in X such that, for each $n \in \Gamma^+$, $gx_{2n} \in Tx_{2n-1}$ and $fx_{2n+1} \in Sx_{2n}$. Using same argument of Theorem 3, $\{fx_{2n+1}\}$ and $\{gx_{2n}\}$ are Cauchy sequences in X and so, since (X, d) is a complete metric space, they converges to some points in X and their limits are equal.

Let $\lim_{n \to \infty} fx_{2n+1} = \lim_{n \to \infty} gx_{2n} = z$. Since f^t and g^s are contikuous,

$$\begin{aligned} &f^tgx_{2n} \longrightarrow f^tz, \\ &f^tfx_{2n-1} \longrightarrow f^tz, \\ &g^sfx_{2n+1} \longrightarrow gz, \\ \text{and} \quad &g^sgx_{2n} \longrightarrow gz \text{ as } n \longrightarrow \infty. \end{aligned}$$

By (4) and (5), we can also show that z is a coincidence point of f, g, S and T and

$f^t z(=g^s z)$ is a common fixed point of f, g, S and T.

Corllary 6. *Let f and g be continuous mappings from X into itself.* Suppose that $\mathcal{J}$ is a family of multivalued mappings from X into $C(X)$ such that

(16) $Ffx=fFx$ *and* $Fgx=gFx$ *for all* $x\in X$ *and* $F\in\mathcal{J}$,

(17) $F(f^t(X))\subseteqq g^s(X)$ *and* $F(g^{s-1}(X))\subseteqq f^t(X)$ for some fixed positive integers s and t, If for all s, $T\in\mathcal{J}$, (5) of Theorem 3 holds, then f, g and $\mathcal{J}$ have a common fixed point in X.

Remark 3. *Corollary 6 extends the main theorens of [2] and [4] to multivalued mappings.*

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