韓國數學教育學會誌〈數學教育〉
1987. 12. Vol. XXVI, No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1987. Vol. XXVI, No. 1

The existence of a lower solution of $\frac{4(n-1)}{n-2}\Delta u+Ku^{\frac{n+2}{n-2}}=0$ on compact manifolds

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1. Introduction

A basic problem in Riemannian geometry is that of studying the set of curvature functions that a manifold possesses. In this generality there has been such a great deal of work [2, 3, 4, 5]. However, in this paper we shall be concerned with the existence of a solution of

$$(1.1)\quad \frac{4(n-1)}{n-2}\Delta u + Ku^{\frac{n+2}{n-2}} = 0,\quad u>0.$$

Let (M, g) be a Riemannian manifold of dimension n and K be a given function on M. One may ask the following question : can we find a new metric g_1 on M such that K is the scalar curvature of g_1 and g_1 is conformal to g(i.e., there exists $u>0$ on M such that $g_1 = u^{\frac{4}{n-2}} g$)?

If M admits $k\equiv 0$ as the scalar curvature of g, then this is equivalent to the problem of solving the elliptic equation

$$\frac{4(n-1)}{n-2}\Delta u+Ku^{\frac{n+2}{n-2}}=0,\quad u>0,$$

where Δ is the Laplacian in the g metric.

In [4], J.L.Kazdan and F.W.Warner have studied the necessary conditions of the solvability of (1.1), i.e., K changes sign and $\bar{K}<0$. In this paper we shall prove the existence of a lower solution of (1.1).

2. Main result

Let (M, g) be a compact connected manifold of dimension n, which is not necessarily orientable. We denote the volume element of this metric by dV, the gradient by ∇, and the mean value of a function f on M is written $\bar{f}$, that is,

$$\bar{f} = \frac{1}{vol(M)}\int_M f\, dV.$$

We let $H_{s,p}(M)$ denote the Sobolev space of functions on M whose derivatives through order s are in L_p. The norm on $H_{s,p}(M)$ will be denoted by $\|\cdot\|_{s,p}$. The usual $L_2(M)$ inner product will be written $<,>$.

It turns out that (1.1) is easier to analyze if we free it from geometry and consider instead

(2.1) $\Delta u + Hu^a = 0,\ u > 0,$

where H is an arbitrary function and $a > 1$ is a constant.

Lemma 1. *Let $\dim M \geq 3$ and $p > \dim M$. Then there exists a constant $C > 0$ such that for any $u \in H_{1,p}(M)$, $\|u\|_\infty \leq C\|u\|_{1,p}$.*

Proof. See 2.22 in [5] or equation (3.8) in [3].

Lemma 2. *If a positive solution u of (2.1) exists and $H \not\equiv 0$, then H must change sign and $\bar{H} < 0$.*

Proof. See Lemma 2.5 and Prop. 5.3 in [4].

Theorem. *(Existence of a lower solution)*

Let $H(\not\equiv 0)$ belong to $C^\infty(M)$ such that H changes sign and $\bar{H} < 0$. Then there exists a lower solution $u > 0$ of (2.1).

Proof. Taking the change of variable $u = e^v$, v satisfies

$$\Delta v + |\nabla v|^2 + He^{cv} = 0,$$

where $c = a - 1 > 0$ is a constant. We claim that there exists $v \in H_{1,p}$ such that $\Delta v + He^{cv} = 0$. For this claim, define a set of functions B by $B = \{v \in H_{1,p}(M) : \int_M He^{cv}\,dV = 0, \bar{v} = 0\}$.

Since H changes sign, it is easy to see that B is not empty. We shall minimize the functional

$$J(v) = \int_M |\nabla v|^2\,dV = \|\nabla v\|_2^2 \quad for\ v \in B.$$

Clearly $J \geq 0$. Let $b = inf_{v \in B} J(v)$. Say $\{v_n\} \in B$ is a minimizing sequence, so $J(v_n) \downarrow b$. Because B is not empty, there is some $v_0 \in B$. Let $b_1 = J(v_0)$. Then we can assume $J(v_n) \leq b_1$ for all n. Since M is compact, the Hölder inequality implies that $v_n \in H_{1,2}(M)$ for each n. But $\bar{v}_n = 0$, so the Poincare inequality also implies that $\|v_n\|_{1,2}^2 \leq$ constant $\times J(v_n) \leq$ constant for all n. Because the unit ball in any Hilbert space is weakly compact, we conclude that there is some $v \in H_{1,2}(M)$ such that a subsequence of the v_n's, which we relabel v_n, converges weakly to v. This implies that $\bar{v} = 0$).

Since $H_{1,2}(M) \subsetneq L_2(M)$ is compact(the Kondrakov's imbedding theorem there is some $v_0 \in L_2(M)$ such that v_{n_k} converges strongly to v_0. So $v_{n_k} \to v_0$ weakly in $L_2(M)$, i.e., $v_0 = v$. For each n, $v_n \in H_{1,p}$. Let $\tilde{a} = inf_{v_n}\|v_n\|_{1,p}$. There exists a subsequence $\{v_{n_k}\}$ of v_n's such that $\|v_{n_k}\|_{1,p} \to \tilde{a}$. Let $b_2 = \|v_{n_0}\|_{1,p}$. Then we may assume that $\|v_{n_k}\|_{1,p} \leq b_2$ for all n_k. Because $H_{1,p}$ is reflexive, the unit ball in $H_{1,p}$ is weakly sequentially compact, so we conclude that there is some $\tilde{v} \in H_{1,p}$ such that v_{n_k} converges $\tilde{v}$ weakly.

Since $H_{1,p} \subsetneq C^\alpha$ is compact for some small $\alpha > 0$, there is some $\tilde{v}_0 \in C^\alpha(M)$ such that a subsequence of the v_{n_k}'s, which we relabel v_{n_k}, converges strongly to $\tilde{v}_o$. But $\|v_{n_k}-\tilde{v}_o\|_p\leq\|v_{n_k}-v_o\|_\infty\leq\|v_{n_k}-v_o\|_{C^\alpha}$ so $v^{n_k}\to\tilde{v}_0$ strongly in L_p. There $\tilde{v}_o=\tilde{v}$. Since $\{v_{n_k}\}$ is a subsequence of v_n's and $\|v_{n_k}-\tilde{v}_o\|_2\leq constant\times\|v_{n_k}-\tilde{v}_o\|_p$, $v_{n_k}\to\tilde{v}_o$ strongly in $L_2(M)$. Hence $v=v_o=\tilde{v}=\tilde{v}_{o}$, i. e., $v\in H_{1,p}\cap\tilde{H}_{1,2}$.

Combining these facts and using the inequality $|e^t-1|\leq|t|e^{|t|}$, we find that

$$
\begin{aligned}
|\textstyle\int_M He^{cv}| &= |\textstyle\int_M(He^{cv}-He^{cv_n})| \\
&\leq \textstyle\int_M|H|\,|e^{cv}-e^{cv_n}| \\
&\leq \|H\|_\infty\textstyle\int_M e^{cv}|1-e^{cv_n-cv}| \\
&\leq \|H\|_\infty e^{c\|v\|_\infty}e^{\|cv_n-cv\|_\infty}\|cv-cv_n\|_{C^\alpha}\to 0
\end{aligned}
$$

Since $\|v_n-v\|_{C^\alpha}\to 0$.

Hence $\int_M He^{cv}\,dV=0$ *and* $\bar{v}=0$, *i. e.,* $v\in B$.

To conclude that v minimizes J for all $v\in B$, we use the general result that whenever v_n converge to v weakly in a normed space, $\|v\|\leq liminf\ \|v_n\|$. (See theorem 3.17 in 5) Hence $\|v\|_{1,2}<lim\ inf\|v_n\|_{1,2}$ Since $\tilde{H}_{1,2}(M)=\{v\in H_{1,2}(M):\bar{v}=o\}$ is a Hilbert subspace, $\sqrt{J(v)}$ is a norm equivalent to the norm $\|\cdot\|_{1,2}$ *on* $\tilde{H}_{1,2}(M)$. Therefore $J(v)\leq J(v_n)$ for all n. Thus v minimizes J in B.

Since v minimizes J in B, by the standard Langrange multiplier theory, we find that there are constants and such that for any $\varphi\in H_{1,p}(M)$.

$$\int_M[2\nabla v\nabla\varphi+\lambda He^{cv}\varphi+\mu\varphi]\,dV=0.$$

This is the Euler-Langrange equation. Since $H\in C^\infty(M)$, by Lp regularity theory, $v\in C^\infty(M)$. $\varphi\equiv 1$ gives $\mu=0$. And $\varphi=e^{-cv}$ and $\bar{H}<0$ show that $\lambda<0$. So we can write $-\lambda=2e^r$ for some constant r. Then $u=v+r$ is the desired solution $u\in C^\infty(M)$ of $\triangle u+He^{cu}=0$. 0. Thus u is the lower solution of $\triangle u+Hu^a=0$, $u>0$.

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