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Minimax Theorems for Set Functions

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A useful concept is that of optimization depending on parameters; minimizing $F(\Omega, u)$ over all $\Omega \in \alpha$. The parameter u ranges over a space U. For example, u might be time, or it might be a random vector expressing uncertainty in the data defining the problem. Or, u might simply represent certain variables whose effect on the problem is of interest.

In the first section of this paper, we define a convex set function on $C \times U$, where c is a convex subfamily of a σ-algebra α, and U is a linear space. Also, we think of some properties of such functions. In section 2, we will modify minimax theorems of Ky Fan [3] to set functions in paired spaces.

1. Set Functions in a Paired Space

We begin with a definition of convex set function in a paired space, which will be used in parametric optimization problem.

Definition 1. Let c be a convex subfamily of a σ-algebra α and let u be a linear space. A set function F on $C \times U$ is said to be convex if and only if for any given (Ω, u), $(\Lambda, v) \in C \times U$, $\lambda \in I$ and Morris-sequence associated with $\langle \lambda, \Omega, \Lambda \rangle$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ satisfying

$$\lim_{k \to \infty} \sup F(\Gamma_{n_k}, u + (1-\lambda) V) \leq \lambda F(\Omega, u) + (1-\lambda) F(\Lambda, V).$$

Definition 2. The epigraph of F, $[F: C \times U]$, is convex in $C \times U \times R$ if, for given (Ω, u, r), $(\Lambda, v, s) \in [F: C \times U]$, $\lambda \in I$ *and* Morris-sequence $\{\Gamma_n\}$ associated with $\langle \lambda, \Omega, \Lambda \rangle$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ and a sequence $\{t_k\}$ satisfying

$$t_k \to \lambda r + (1-\lambda) s \text{ and } F(\Gamma_{n_k}, \lambda u + (1-\lambda) v) \leq t_k.$$

Remark: *(i) It is elementary to verify that F is convex in $C \times U$ if and only if the epigraph of F, $[F: C \times U]$, is convex in $C \times U \times R$.*
(ii) It is also easy to show that the convexity of F in a paired space $C \times U$ implies the convexity of F in C for each $u \in U$ and the convexity of F in U for each $\Omega \in C$.

The following theorem shows that the function $\Phi(u) = \inf_{\Omega \in C} F(\Omega, u)$, which is called the optimal-value function, is convex if F is convex in $C \times U$.

Theorem 1. *Let C be a convex subfamily of α, and let U be a linear space. If F is a* *convex set function in $C\times U$, then the optimal-value function $\Phi(u)=\inf_{\Omega\in C} F(\Omega,u)$ is convex on U.*

Proof: For given $u, v\in U$, let $\Phi(u)=r$ and $\Phi(v)=s$. Then, $\inf_{\Omega\in C} F(\Omega,u)=r$ and $\inf_{\Omega\in C} F(\Omega,v)=s$ imply the existence of sequences $\{\Omega_n\}$, $\{\Lambda_m\}$ in c and sequences $\{r_n\}$, $\{s_m\}$ in R, such that $F(\Omega_n, u)=r_n$ with $r_n\to r$ and $F(\Lambda_m, V)=s_m$ with $s_m\to s$. For fixed n, m, consider (Ω_n, u, r_n) and (Λ_m, V, S_m), which belong to $[F:C\times U]$, Let $\{\Gamma_j^{(n,m)}\}$ be a Morris-sequence associated with $\langle\lambda, \Omega_n, \Lambda_m\rangle$. Since F is convex, there exists a subsequence $\{\Gamma_{jk}^{(n,m)}\}$ of $\{\Gamma_j^{(n,m)}\}$ and a sequence $\{t_k^{(n,m)}\}$ which converges to $\lambda r_n+(1-\lambda)s_m$ such that

$$F(\Gamma_{jk}^{(n,m)}, \ \lambda u+(1-\lambda)v) \leqq t_k^{(n,m)}$$

Therefore, for all k,

$$\inf_{\Omega\in C} F(\Omega, \ \lambda u+(1-\lambda)v) \leqq t_k^{(n,m)}$$

Since $\quad t_k^{(n,m)}\to\lambda r_n+(1-\lambda)s_m$ as $k\to\infty$,

$$\inf_{\Omega\in C} F(\Omega, \lambda u+(1-\lambda)v) \leqq \lambda r_n+(1-\lambda)s_m.$$

The above inequality holds for every n and m. Therefore, by letting $n, m\to\infty$,

$$\inf_{\Omega\in C} F(\Omega, \lambda u+(1-\lambda)v) \leqq \lambda r+(1-\lambda)s.$$

Hence,

$$\Phi(\lambda u+(1-\lambda)v) \leqq \lambda r+(1-\lambda)s \leqq \lambda\Phi(u)+(1-\lambda)\Phi(v).$$

Thus, we have shown that Φ is convex on U.

2. Minimax Theorems of Ky Fan Type

Without the structure of lineality, Ky Fan [3] generalized the definition of convex function and established minimax theorems for functions defined on the product space X×Y of two arbitrary sets X and Y. In this section, we modify his results for convex set functions on the product space $\alpha\times\beta$, where α and β are σ-algebras. Note that the definition of convexity in set functions is different from that used by Ky Fan.

Theorem 2. *Let α and β be σ-algebras, and let F be a set function $\alpha\times\beta$ such that for every $\Lambda\in\beta$, $F(\Omega,\Lambda)$ is $w^*-1.s.c.$ on α. If F is convex on α and concave on β, then*

$$\inf_{\Omega\in\alpha}\sup_{\Lambda\in\beta} F(\Omega,\Lambda)=\sup_{\Lambda\in\beta}\inf_{\Omega\in\alpha} F(\Omega,\Lambda). \tag{1}$$

Proof: For fixed $\Lambda\in\beta$, consider the $w^*-1.s.c.$ extension $\bar{F}$ on $\bar{\alpha}\times\beta$, i.e.,

$$F(f,\Lambda)=\lim_{V\in N(f)}\inf_{\Omega\in V\cap\alpha} F(\Omega,\Lambda), \quad f\in\alpha$$

Then, as shown in [2], α is a compact subset of the L_∞-space and F is convex on $\bar{\alpha}$ for each $\Lambda\in\beta$. Now, for fixed $f\in\bar{\alpha}$, we shall show that F is concave on β. Let $\lambda\in I$ and Λ_1, $\Lambda_2\in\beta$ be given. For a given Morris-sequence $\{\Gamma_n\}$ associated with $\langle\lambda, \Lambda_1, \Lambda_2\rangle$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ such that

$$\liminf_{k\to\infty} F(\Omega,\Gamma_{n_k})\geqq\lambda F(\Omega,\Lambda_1)+(1-\lambda)\ F(\Omega,\Lambda_2), \quad \Omega\in\alpha. \tag{2}$$

By the definition of $\bar{F}$

$$\lambda\bar{F}(f,\Lambda_1)+(1-\lambda)\ \bar{F}(f,\Lambda_2)$$
$$\leqq\lim_{V\in N(f)}\inf_{\Omega\in V\cap\alpha}\{\lambda F(\Omega,\Lambda_1)+(1-\lambda)\ F(\Omega,\Lambda_2)\}. \tag{3}$$

Using (2),

$$\lim_{V\in N(f)}\inf_{\Omega\in V\cap\alpha}\{\lambda F(\Omega,\Lambda_1)+(1-\lambda)\ F(\Omega,\Lambda_2)\}$$

$$\leq \lim_{V\in N(f)} \inf_{\Omega\in V\cap\alpha} \{\liminf_{k\to\infty} F(\Omega, \Gamma_{n_k})\}$$
$$= \liminf_{k\to\infty} \{\lim_{V\in N(f)} \inf_{\Omega\in V\cap\alpha} F(\Omega, \Gamma_{n_k})\}$$
$$= \liminf_{k\to\infty} \bar{F}(f, \Gamma_{n_k}).$$

By (3) and (4),

$$\liminf_{k\to\infty} \bar{F}(f, \Gamma_{n_k}) \geqq \lambda\bar{F}(f, \Lambda_1) + (1-\lambda)\ \bar{F}(f, \Lambda_2).$$

Hence, $\bar{F}$ concave on β for each $f \in \bar{\alpha}$. Now, we can apply the same procedure as in Theorem 2 [3] to prove (1) without any difficulty.

Another minimax theorem which is free of w^*-lower semicontinuity is made possible by using almost periodic functions. A generalized definition of almost periodic function was introduced by Ky Fan.
We begin with a definition of almost periodic set function.

Definition 3. Let α and β be two σ-algebras. A set function F on $\alpha\times\beta$ is said to be right almost periodic if F is bounded on $\alpha\times\beta$ and if, for arbitrary $\varepsilon>0$, there exists a finite covering $\beta=\bigcup_{k=1}^{m} \beta_k$ of β such that

$$| F(\Omega, \Lambda') - F(\Omega, \Lambda'') | < \varepsilon$$

for all $\Omega\in \alpha$, whenever Λ', Λ'' belong to the same β_k. Left almost periodic set functions are defined similarly.

Lemma 3. *(i) Every right almost periodic set function is also left almost periodic and vice versa. Thus we simply use the term almost periodic.*
(ii) Let X, $Y\subset R$ and let α and β be σ-algebras. Let F_1: $\alpha \to X$ and F_2: $\beta \to Y$ be arbitrary set functions. If u: $X\times Y \to R$ is an almost periodic function, then the set function G: $\alpha\times\beta \to R$ defined by

$$G(\Omega, \Lambda) = u(F_1(\Omega),\ F_2(\Lambda))$$

is almost periodic.

Proof: (i) See [3, p 47].
(ii) By (i), it is enough to show that G is right almost periodic. For any $\varepsilon>0$, let $\bigcup_{k=1}^{m} Y_k = Y$ be a finite covering of Y with

$$| u(x, y') - u(x, y'') | < \varepsilon \text{ whenever } y', y''\in Y_k.$$

Let $\beta_k = \{\Lambda\in\beta:\ F_2(\Lambda)\in Y_k\}$. Then $\bigcup_{k=1}^{m} \beta_k=\beta$.
Furthermore, for Λ', $\Lambda''\in \beta_k$,

$$| G(\Omega, \Lambda') - G(\Omega, \Lambda'') |$$
$$= | u(F_1(\Omega),\ F(\Lambda')) - u(F_1(\Omega),\ F_2(\Lambda'')) | < \varepsilon,$$

because $F_1(\Lambda')$, $F_2(\Lambda'')$ belong to the same *set* Y_k. Since the boundedness of G is clear, G is right almost periodic.

Using the above lemma, many examples of almost periodic functions can be constructed easily. The next minimax theorem is free of w^*-lower semicontinuity, and is also a modification of Ky Fan's theorem for set functions.

Theorem 4. *Let F be an almost periodic set function on the product set $\alpha\times\beta$, where α and β are σ-algebras. Then:*

(i) The equality

$$\inf_{\Omega\in\alpha}\sup_{\Lambda\in\beta} F(\Omega,\Lambda)=\sup_{\Lambda\in\beta}\inf_{\Omega\in\alpha} F(\Omega,\Lambda) \tag{5}$$

holds if and only if the following condition is satisfied: For any $\varepsilon>0$ and any two finite sets $\{\Omega_1,\cdots,\Omega_n\}\subset\alpha$ and $\{\Lambda_1,\cdots,\Lambda_m\}\subset\beta$, there exist $\Omega_0\in\alpha$, $\Lambda_0\in\beta$ such that

$$|F(\Omega_0,\Lambda_k)-F(\Omega_i,\Lambda_0)|\leqq\varepsilon.\ (1\leqq i\leqq n,\ \ 1\leqq k\leqq m) \tag{6}$$

(ii) In particular, if F is convex on α and concave on β, then (6) holds.

Proof: We proof only that condition (6) implies (5). Everything else is the same as the proof in [3, pp. 46-47]. Let $\{\Omega_1,\cdots,\Omega_n\}\subset\alpha$ and $\{\Lambda_1,\cdots,\Lambda_m\}\subset\beta$ be given. By the minimax theorem in [7, pp. 153-155], there exist two sets $\{\zeta_1,\zeta_2,\cdots,\zeta_n\}$, $\{\eta_1,\eta_2,\cdots,\eta_m\}$ of nonnegative numbers with $\sum_{i=1}^{n}\zeta_i=1$, $\sum_{k=1}^{m}\eta_k=1$ such that

$$\max_{1\leq k\leq m}\sum_{i=1}^{n}\zeta_i F(\Omega_i,\Lambda_k)\leqq\min_{1\leq i\leq n}\sum_{k=1}^{m}\eta_k F(\Omega_i,\Lambda_k). \tag{7}$$

Since $F(\ ,\Lambda_1)$ is convex an α, for any given Morris-sequence $\{\Gamma_n^{(1,1)}\}$ associated with $\langle\lambda,\Omega_1,\Omega_2\rangle$, $\lambda_1\in I$, there exists a subsequence $\{\Gamma_{n_j}^{(1,1)}\}$ *of* $\{\Gamma_n^{(1,1)}\}$ such that

$$\limsup_{j\to\infty} F(\Gamma_{n_j}^{(1,1)},\Lambda_1)\ \leqq\ \lambda_1 F(\Omega_1,\Lambda_1)+(1-\lambda)\ F(\Omega_2,\Lambda_1).$$

Now, consider the Morris sequence $\{\Gamma_n^{(1,2)}\}=\{\Gamma_{n_j}^{(1,1)}\}$. Then the convexity of $F(\cdot,\Lambda_2)$ on α implies the existence of a subsequence $\{\Gamma_{n_j}^{(1,2)}\}$ *of* $\{\Gamma_n^{(1,2)}\}$, actually a subsequence of $\{\Gamma_n^{(1,1)}\}$, satisfying

$$\limsup_{j\to\infty} F(\Gamma_{n_j}^{(1,2)},\Lambda_2)\ \leqq\ \lambda_1 F(\Omega_1,\Lambda_k)+(1-\lambda_1)\ F(\Omega_2,\Lambda_k),$$

for $k=1,2$. We continue this process m times, obtaining a subsequence $\{\Gamma_{n_j}^1\}$ *of* $\{\Gamma_n^{(1,1)}\}$, such that

$$\limsup_{j\to\infty} F(\Gamma_{n_j}^1,\Lambda_k)\ \leqq\ \lambda_1 F(\Omega_1,\Lambda_k)+(1-\lambda_1)\ F(\Omega_2,\Lambda_k),$$

for $k=1,2,\cdots,m$. Therefore, for given $\varepsilon>0$, there exists $\Gamma_o^1\in\alpha$ such that

$$F(\Gamma_o^1,\Lambda_k)-\varepsilon\leqq\lambda_1 F(\Omega_1,\Lambda_k)+(1-\lambda)\ F(\Omega_2,\Lambda_k),\ \ k=1,\cdots,m.$$

Next, consider a Morris-sequence $\{\Gamma_n^{(2,1)}\}$ associated with $\langle\lambda_2,\Gamma_o^1,\Omega_3\rangle$, $\lambda_2\in I$. Since $F(\cdot,\Lambda_1)$ is convex on α, there exists a subsequence $\{\Gamma_{n_j}^{(2,1)}\}$ such that

$$\limsup_{j\to\infty} F(\Gamma_{n_j}^{(2,1)},\Lambda_1)\ \leqq\ \lambda_2 F(\Gamma_o^1,\Lambda_1)+(1-\lambda_2)\ F(\Omega_3,\Lambda_1).$$

Use a similar argument to obtain a subsequence $\{\Gamma_{n_j}^2\}$ of $\{\Gamma_n^{(2,1)}\}$, such that

$$\limsup_{j\to\infty} F(\Gamma_{n_j}^2,\Lambda_k)\ \leqq\ \lambda_2 F(\Gamma_o^1,\Lambda_k)+(1-\lambda_2)\ F(\Omega_3,\Lambda_k),$$

for $k=1,2,\cdots,m$. Hence for any given $\varepsilon>0$, there extsts $\Gamma_o^2\in\alpha$ such that

$$\begin{aligned}F(\Gamma_o^2,\Lambda_k)-\varepsilon&\leqq\lambda_2 F(\Gamma_o^1,\Lambda_k)+(1-\lambda_2)\ F(\Omega_3,\Lambda_k)\\&\leqq\lambda_2\lambda_1 F(\Omega_1,\Lambda_k)+\lambda_2(1-\lambda_1)\ F(\Omega_2,\Lambda_k)\\&\qquad+(1-\lambda_2)\ F(\Omega_3,\Lambda_k)+\lambda_2\varepsilon,\ \ k=1,2,\cdots,m.\end{aligned}$$

Note that $\lambda_2\lambda_1+\lambda_2(1-\lambda_1)+(1-\lambda_2)=1$ and $\lambda_2\lambda_1$, $\lambda_2(1-\lambda_1)$, $1-\lambda_2$ are all nonnegative. Continuing this process for all the Ω_i's$(i=1,\cdots,n)$, we have the following result.

For any given $\varepsilon>0$, there is $\Omega_o\in\alpha$ such that

$$F(\Omega_o,\Lambda_k)\leqq\zeta_1 F(\Omega_1,\Lambda_k)+\ \cdots\ +\zeta_n F(\Omega_n,\Lambda_k)+(n-1)\,\varepsilon,\ \ k=1,\cdots,m. \tag{8}$$

Since the λ_i's, $i=1,\cdots,n-1$, are arbitrary real numbers in I, appropriate values can be chosen so that the coefficients of $F(\Omega_i,\Lambda_k)$ are ζ_i's, $i=1,\cdots,n$.

In the same manner, the concavity of $F(\Omega,\cdot)$ on β for each $\Omega\in\alpha$ implies the following: for any given $\varepsilon>0$ there is $\Lambda_o\in\beta$ such that

$$F(\Omega_i,\Lambda_o)\geqq\eta_1 F(\Omega_i,\Lambda_1)+\cdots+\eta_m F(\Omega_i,\Lambda_m)+(m-1)\,\varepsilon,\ \ i=1,\cdots,n. \tag{9}$$

Combine (7), (8) and (9) to obtain

$$\begin{aligned} F(\Omega_o, \Lambda_k) &\leqq \sum_{i=1}^{n} \zeta_i F(\Omega_i, \Lambda_k) + (n-1)\varepsilon \\ &\leqq \sum_{k=1}^{m} \eta_k F(\Omega_i, \Lambda_k) + (n-1)\varepsilon \\ &\leqq F(\Omega_i, \Lambda_o) + (m+n-2)\varepsilon. \ (1 \leqq i \leqq n, \ 1 \leqq k \leqq m) \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete.

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