Multi-Stage Production-Inventory Planning for Deteriorating Items

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要 旨

多段階 生產-在庫문제에 관하여 일반적인 해법으로서 Dynamic Programming이 이용되고 있으나 C. L. Hwang과 L. T. Fan은 Pontryagin의 最大原理에 의해 보다 효율적인 방법을 제시하였다. 본 연구는 상기의 문제를 製品의 陳腐化가 있는 모델로 확장하여 離散型 最大原理를 적용하는 것으로서, 특히 需要 및 陳腐化率이 期間마다 변화하는 多段階 生產-在庫시스템에 있어서의 最適 生產-在庫政策을 수립하는 효율적인 앨고리듬을 제시한다.

Abstract

A Multistage production-inventory model is developed for deteriorating items. The model is developed deterministic but time-varing demand pattern and instantaneous delivery. Deterioration rates are assumed to vary from period to period. Discrete version of Pontryagin's maximum principle is used to present the efficient alogrithm to solve this model easily. A numerical example is given to illustrate the derived results.

1. Introduction

The effect of deterioration is so vital in many iventroy systems that it cannot be disregarded. Here deterioration is defined as decay, demage or spoilage such that the item cannot be used for original purpose. Food items, photo films, drugs and pharmaceuticals, chemicals and radio-active substances are examples of items in which sufficient deterioration may occur during the normal stroge period of the units consequently this less must be taken into account while analyzing the system.

In this paper a multistage production-inventory model is developed under deterministic but time-varing demand pattern and instantaneous delivery. The amount of deterioration during a period is assumed to be a fraction of the on hand inventory. It is further assumed that the deterioration rates very from period to period.

Discrete version of Pontryagin's maximum principle is employed to present the more efficient algorithm than the solution procedure using dynamic programming to obtain an optimal production-inventory policy.

2. Discrete Version of Maximum Principle

The maximum principle is an optimization technique that was first proposed by Pontryagin and his

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associates[1] for various types of time-optimizing continous process. The development of the maximum principle has been thoroughly documented by Fan and his associates[2, 3, 4]. A shematical representation of a simple multistage process is shown in Fig. 1.

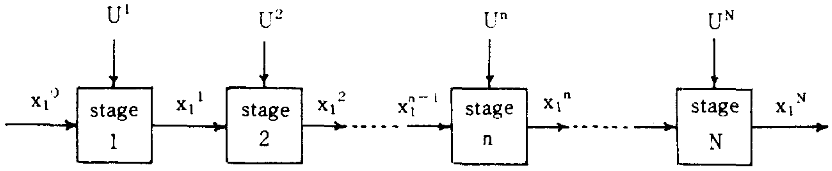

Figure-1. Multistage Decision Process

The process consists of N stages connected in series. The state of the process stream denoted by an s-dimensional vector, $x=(x_1,x_2,\cdots,x_S)$, is transformed at each stage according to an r-dimensional decision vector, $u=(u_1,u_2,\cdots,u_r)$, which represents the decisions made at that stage. The transformation of the process stream at n th stage is described by a set of performance equations in vector form,

$$x^n=T^n(x^{n-1};u^n),\ (n=1,2,\cdots,N) \qquad (1)$$
$$x^0=\alpha$$

A typical optimization problem associated with such a process is to find a sequence of u^n, $n=1,2,\cdots,N$, subject to constraints

$$\phi_i^n\ \{u_1^n,\ u_2^n,\ \cdots,\ u_r^n\}\leqslant 0,\ (n=1,2,\cdots,N\ :\ i=1,2,\cdots,r) \qquad (2)$$

which makes a function of the state variable of the final stage

$$J=\sum_{i=1} C_iX_i^N,\ (C_i=\text{constant}) \qquad (3)$$

an extremum when the initial condition $x^0=\alpha$ is given. The function, J, which is to be maximized (or minimized), is the objective function of the process.

The procedure for solving such an optimization problem by a discrete version of the maximum principle is to introduce an s-dimensional adjoint vector Z^n and a Hamiltonian function H, which satisfy the following relations:

$$H^n=(Z^n)^TX^n=\sum_{i=1} Z_i^nT_i^n(X^{n-1};U^n),\ (n=1,2,\cdots,N) \qquad (4)$$
$$Z_i^{n-1}=\partial H^n/\partial X_i^{n-1},\ (i=1,2,\cdots,S;n=1,2,\cdots,N) \qquad (5)$$
$$\text{and }\ Z_i^n=C_i,\ (i=1,2,\cdots,S)$$

In order for the objective function J to attain its(local) maximum (or minimum) value it is necessary (but not sufficient) to choose a set of decision U^n, $n=1,2,\cdots,N$ such that:

(a) the Hamitonian is made stationary with respect tothe optimal decision when it is not constrained (or it lies in the interior of the admissible domain of u), that is

$$\bar{U}^n:\ (\partial H^n/\partial u^n)=(Z^n)^T(\partial T^n/\partial u^n)=0,\ (n=1,2,\cdots,N) \qquad (6)$$

or (b) the Hamiltonian is made a (local) maximum (or minimum) with respect to u^n when it lies on a constraint, that is

$$\bar{U}^n\ :\ H^n=\text{maximum(minimum)},\ (n=1,2,\cdots,N) \qquad (7)$$

3. Multiperiod Production-Inventory System for Dateriorating Item

Although some researchers feel that the dynamic inventory problem has been completely soved, there is one very important aspect to the problem which has little attention in the literature; specifically, optimal ordering policies for deteriorating items. Wagner and Whitin[5] proposed a dynamic version of economic lot size model and Freidman and Hoch[6] and Veinott[7] developed a model similar to them considering the constant deterioratin rate. Recently Choi and Kim[8] proposed the dynamic inventory model for deteriorrating items with the time-varing deterioration rates and solved this problem using dynamic programming.

The purpose of this paper is to present an optimal production-inventory policy for deteriorating items by means of the discrete version of Pontryagin's maximum principle.

3.1 Model Formulation

The model is developed under the following assumptions:

1. A finite horizons is devided into N(time unit $n=1,2,\cdots,N$) periods.
2. Demand and replenishment are occured at the beginning of each periods.
3. Deliveries are instanteous.
4. The deteriorated items in the inventory are neither repaired nor replaced.
5. Deterioration rates are represented the fraction of the on hand inventory of each periods and can vary from period to period.
6. Demand pattern is deterministic but time-varing.

The notations used in this model is as follows:

S^n=amount of demand in period n

I^n=inventory level at the end of period in

P^n=amount of production in period n

θ^n=deterioration rate in period in

V^n=selling price per unit in period n

R^n=amount of sales in period n

C =fundmental cost concerned with production and inventory in each periods

C^n=cost in period n

G^n=profit in period n

In a manufacturing company, forecasting is used in designing production rules which anticipate and prepare for sales fluctuations. Specially in the case of deteriorating items a buffer inventory is maintained so that errors in sales forecosts will not runouts or will not force rapid changes in the rate of plant operation.

For the multiperiod production-inventory system considering the deteriorating, as shown in Figure-2, there is the relationship among the inventory level, amount of production and amount of sales as follows:

$$I^n=(I^{n-1}+P^n-S^n)(1-\theta^n),\ (n=1,2,\cdots,N) \tag{8}$$

Here we assume that the selling price of item vary from period to period according to the diffence between the amount of production and sales. Thus the selling price in period n is

$$V^n=V+C_V(S^n-P^n),\ (n=1,2,\cdots,N) \tag{9}$$

where V and C_V are constants. Then the total sales in period n is

$$R^n=S^n\{V+C_V(S^n-P^n)\},\ (n=1,2,\cdots,N) \tag{10}$$

On the other hand, we assume that the rate of costs from holding inventories, stockouts and deteriortions can be approximated by the quadratic $C_I(I^n - I^*)$, and the rate at which manufacturing cost are incurred can be approximated by the quadratic $C_P(P^n - P^*)$, where C_I and C_P are constants, and I^* and P^* represent the desired inventory and production level of the plant, respectively. Then the total cost in period n is

$$C^n = C + C_I(I^n - I^*)^2 + C_P(P^n - P^*)^2, \ (n=1,2,\cdots,N) \tag{11}$$

where C is constant.
From (10) and (11), the profit in period n is

$$G^n = R^n - C^n, \ (n=1,2,\cdots,N) \tag{12}$$

Therefore the total profit from period 1 to period N is

$$J = \sum_{n=1}^{N} G^n = \sum_{n=1}^{N} [S^n \{V + C_V(S^n - P^n)\} - C - C_I(I^n - I^*)^2 - C_P(P^n - P^*)^2] \tag{13}$$

and it is the objective function of this problem.

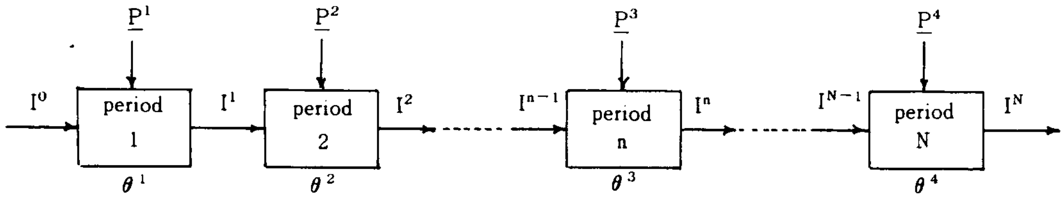

Figure-2. Multiperiod Production-Inventory System for Deteriorating Items

3.2 Optimal Production-Inventory Planning

Now we let

$$I^n = X_1^n, \ P^n = U^n$$

Then, from (1) and (8), we obtain

$$\begin{aligned} &X_1^n = T_1^n(X_1^{n-1}, U^n) = (X_1^{n-1} + U^n - S^n)(1 - \theta^n), \\ &X_1^0 = I^0 (\text{initial inventory level}), \end{aligned} \tag{14}$$

and the objective function J is

$$J = \sum_{n=1}^{N} [S^n \{V + C_V(S^n - U^n)\} - C - C_I(X_1^n - I^*)^2 - C_P(U^n - P^*)^2]. \tag{15}$$

Let a new state variable X_2^n be the profit from period 1 to period n, then

$$\begin{aligned} X^n &= T_2^n(X_2^{n-1}, U^n) = X_2^{n-1} + G^n \\ &= X_2^{n-1} + S^n \{V + C_V(S^n - U^n)\} - C - C_I \{(X_1^{n-1} + U^n - S^n)(1 - \theta^n) - I^*\} - C_P(U^n - P^*)^2 \end{aligned} \tag{16}$$

From (3), the objective function is

$$J = \sum_{i=1}^{2} C_i X_i^N = C_1 X_1^N + C_2 X_2^N = X_2^N. \tag{17}$$

Hence $c_1 = 0$ and $c_2 = 1$.
To solve this problem, we can introduce the adjoint vector Z^n and the Hamiltonian function H^n which satisfy the following relations:

$$H^n = \sum_{i=1}^{2} Z_i^n T_i^n$$

$$= Z_1^n \{(X_1^{n-1} + U^n - S^n)(1-\theta^2)\} + Z_2^n [X_2^{n-1} + Sn \{V + C_V(S^n - U^n)\} - C - C_I \{(X_1^{n-1} + U^n - S^n)(1-\theta^n) - I^*\}^2 - C_P(U^n - P^*)^2], \quad (18)$$

$$Z_1^{n-1} = \partial H^n / \partial X^{n-1} = Z_1^n(1-\theta^n) + Z_2^n \{-2C_I(X_1^{n-1} + U^n - S^n)(1-\theta^n) - I^*\}(1-\theta^n), \quad (19)$$

$$Z_2^{n-1} = \partial H^n / \partial X_2^{n-1} = Z_2^n.$$

The boundary condition of the adjoint equation is

$$Z_1^N = \partial J / \partial X_1^N = 0, \quad Z_2^N = \partial J / \partial X_2^N = 1. \quad (20)$$

By (6), the necessary condition for maximizing J is

$$\partial H^n / \partial U^n = (1-\theta^n) Z_1^n + Z_2^n [-C_V S^n - 2C_I \{(X_1^{n-1} + U^n - S^n)(1-\theta^n) - I^*\}(1-\theta^n) - 2C_P(U^n - P^*)]. \quad (21)$$

From (19) and (20), we can obtain $Z_2^n = 1$, and derive u^n using (21).

$$U^n = \frac{Z_1^{n-1} - C_V S^n}{2C_P} + P^* \quad (22)$$

In practice, for the most of production systems, the production capabilities are bounded. Thus here we consider the lower bound P_{min} and upper bound P_{max}.
Thus we can define SAT[U] as follows:

$$SAT[U^n] = \begin{cases} P_{max} : U^n > P_{max} \\ U^n : P_{min} < U^n < P_{max} \\ P_{min} : U^n < P_{max} \end{cases} \quad (23)$$

Therefore, the Hamiltonion function and adjoint equations become as follows:

$$X_1^n = \{X_1^{n-1} + SAT[\frac{Z_1^{n-1} - C_V S^n}{2C_P} + P^*] - S^n\}(1-\theta^n) \quad (24)$$

initial condition $X_1^0 = I^0$,

$$Z_1^n = \frac{Z_1^{n-1}}{1-\theta^n} + 2C_I \{(X_1^{n-1} + SAT[\frac{Z_1^{n-1} - C_V S^n}{2C_P} + P^*] - S^n)(1-\theta^n) - I^*\} \quad (25)$$

boundary condition $Z_1^N = 0$.

Consequently, we can obtain the optimal production-inventory policies from (22), (23), (24) and (25).

3.3 Solution Procedure

The flowchart of the solution procedure of this model is shown in Figure-3. To illustrate the computational sheme developed, a numerical example is considered. The input data used for this example is shown in Table 1. In this example, we assume that the planning horizon is devided into 4(n=1,2,3,4) periods and $V=100$, $C=5{,}000$ and $C_V = C_X = C_P = 1$.

We also assume that $I^* = 150$, $P^* = 150$ *and* $I^0 = 150$.

Table-1. Demand and Deterioration Rate.

Item \ Period	1	2	3	4
S^n	150	160	150	140
θ^n	0.1	0.1	0.2	0.1

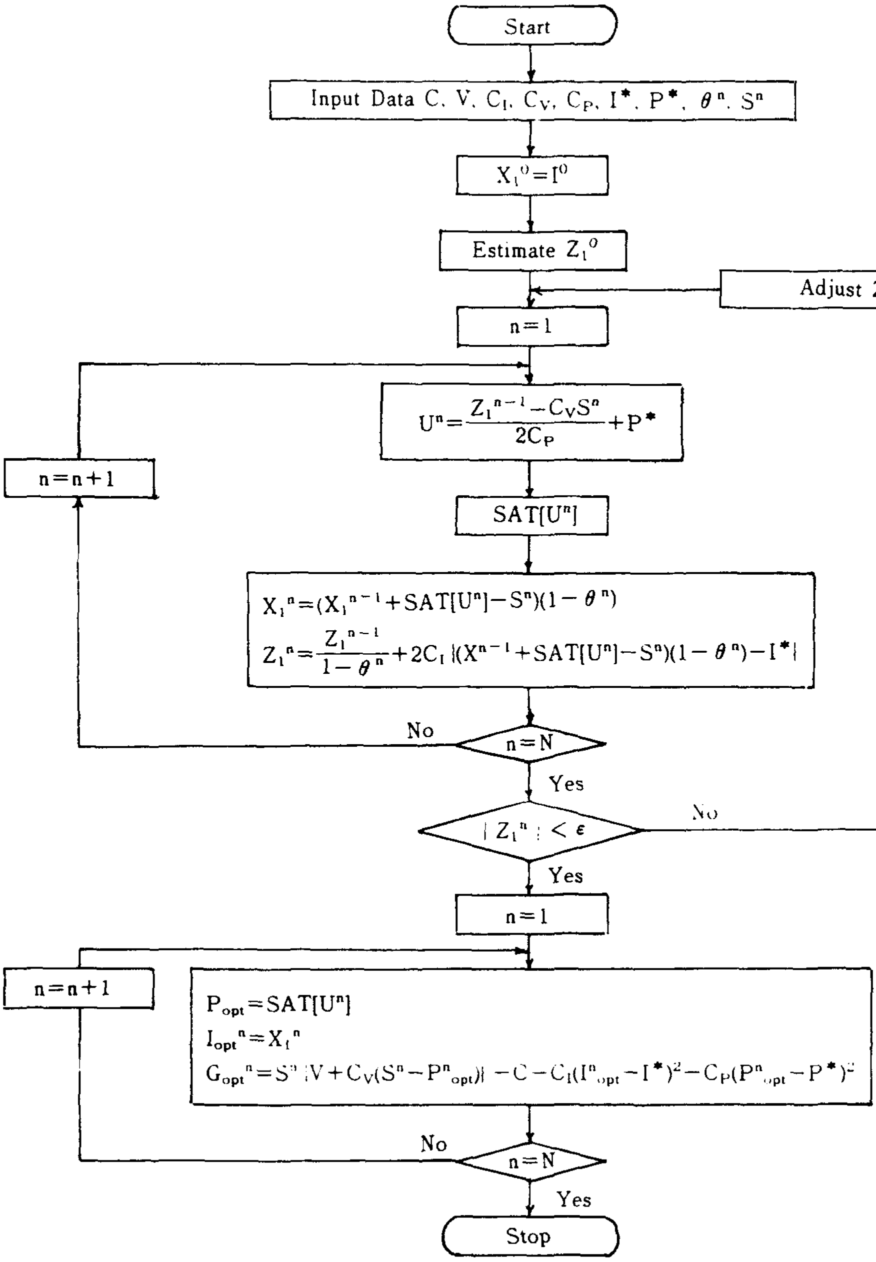

Figure-3. Flowchart of the Solution Procedure.

According to the solution procedure in Figure-3, we obtain the results of computational scheme as shown in Table-2.

Table-2. Results of Computational Scheme

Item \ Period	1	2	3	4
P^n_{opt}	160	159	153	134
I^n_{opt}	144	129	106	90
G^n_{opt}	8.364	10.638	7.605	1.484

4. Conclusion and Extension

Many of the production-inventory systems dealing with food items, chemicals, pertroleum product etc. can be tackled by the present model.

In this paper we drived the results for the optimal policy of multiperiod production-inventory model for deteriorating items. Here we obtain the simple solution procedure by means of the discrete version of the maximum principle.

One important but rather difficult extension is to consider the production leadtime. Another would be to consider the case that the cost functions concered with production and inventory are more efficiently described.

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