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NOTES ON THE PSEUDO-COMPLETE ALGEBRA

AN HYUN KIM

1. Introduction

In [5], Rickart proved that, when F is a Hermitian functional on the Banach $*$-algebra A, in order for F to be representable, it is necessary and sufficient that

(i) F is bounded,
(ii) $|F(x)|^2 \leq \mu F(x^*x)$, $x \in A$

where μ is a positive real constant independent of x. In this note, conditions for a functional to be admissible on a locally convex $*$-algebra are defined and sufficient conditions for a functional F to be representable are also given in Theorem 4.2.

2. Preliminaries

DEFINITION 2.1. By a *locally convex algebra* A we shall mean an algebra A over the complex field C, equipped with a topology τ such that

(i) $(A;\tau)$ is a Hausdorff locally convex topological vector space,

(ii) multiplication is separately continuous.

A will be called a *locally convex $*$-algebra* if A has a continuous involution.

DEFINITION 2.2. Let A be a locally convex algebra. An element x of A is said to be *bounded* if, for some nonzero complex number λ, the set $\{(\lambda x)^n : n \in N\}$ is a bounded subset of A.

The set of all bounded elements of A will be denoted by A_0.

NOTATION. By B_1 we denote the collection of all subsets B of A such that

(i) B is convex and idempotent,

(ii) B is bounded and closed.

If $B \in B_1$, then $A(B)$ will denote the subalgebra of A generated by B, i.e., $A(B) = \{\lambda x : \lambda \in C$ and $x \in B\}$, and the equation $||x||_B = \inf \{\lambda > 0 : x \in \lambda B\}$ defines a norm which makes $A(B)$ a normed algebra.

DEFINITION 2.3. The locally convex algebra A is called *pseudo-complete* if each of the normed algebras $A(B)$ is a Banach algebra.

If A is a locally convex algebra and $x \in A$, we define the *radius of boundedness* of x by

$$\beta(x) = \inf [\lambda > 0 : \{(\lambda^{-1}x)^n : n \in N\} \text{ is bounded}]$$

with the usual convention that $\inf \phi = \infty$.

The following simple facts about $\beta(x)$ are obvious:

1°. $\beta(x) \geq 0$ and $\beta(\lambda x) = |\lambda| \beta(x)$ where $\lambda \in C$ and $0 \cdot \infty = 0$.

2°. $\beta(x) < \infty$ iff $x \in A_0$.

3°. In particular, if A is pseudo-complete, then $\beta(x)$ equals to the spectral radius of x [1].

DEFINITION 2.4. Let A be a locally convex $*$-algebra, and let F be a linear functional on A. If $F(x^*)=(F(x))^-$ for all x in A, then F will be called *Hermitian*. If $F(x^*x)\geq 0$ for all x in A, then F will be called a *positive functional*.

LEMMA 2.5. Let A be a pseudo-complete locally convex $*$-algebra and let x_0 be any element of A such that $\beta(x_0)<1$. Then there exists an element y_0 of A such that $2y_0-y_0^2=x_0$. In addition if x_0 is Hermitian, so is y_0.

PROOF. Consider the function f defined in terms of the binomial series as follows:

$$f(z)=-\sum_{n=1}^{\infty}\binom{1/2}{n}(-z)^n.$$

Then f is well-defined and $2f(z)-[f(z)]^2=z$ for all $|z|\leq 1$. Now consider the vector valued function $-\sum_{n=1}^{\infty}\binom{1/2}{n}(-x_0)^n$.

We show that this series converges. Let $\varepsilon>0$. Since $\beta(x_0)<1$, there exists a $B\in B_1$ by [1] such that $x_0\in A(B)$ and $\|x_0\|_B<1$. Since f converges for $|z|\leq 1$, there exists an n_0 such that for $p, q>n_0$

$$\left\| \sum_{n=p}^{q-1}\binom{1/2}{n}(-x_0)^n \right\|_B <\varepsilon.$$

Since $A(B)$ is complete, we have that vector valued series converges to an element y_0 of $A(B)$ such that $2y_0-y_0^2=x_0$.

THEOREM 2.6. Let A be a pseudo-complete locally convex $*$-algebra and let F be any positive functional on A. Then

$|F(u^*hu)|<\beta(h)F(u^*u)$ for all $u\in A$ and h Hermitian.

PROOF. By Lemma 2.5 and [5, Theorem 4.5.2], the above theorem is obvious.

Let F be a positive functional on A and define

$$L_F=\{x\in A:\ F(y^*x)=0 \text{ for all } y \text{ in } A\}.$$

Then L_F is a left ideal of A([3, p.288]). Now we define $X_F=A/L_F$ and denote $x+L_F$ by $\bar{x}$.

DEFINITION 2.7. A positive linear functional F which satisfies the following conditions will be called *admissible*:

(1) $\sup\{F(x^*a^*ax)/F(x^*x):x\in A\}<\infty$ for all $a\in A$.

(2) For each $x\in A$, there is a $x_0\in A_0$ such that $\bar{x}=\bar{x}_0$.

COROLLARY 2.8. If A is a pseudo-complete locally convex $*$-algebra such that $A=A_0$, then any positive functional is admissible.

PROOF. By Theorem 2.6 and 2°,

$$\{F(x^*a^*ax):x\in A=A_0\}\leq\beta(a^*a)<\infty \text{ for all } a\in A.$$

Since $A=A_0$ for each $x\in A$, there exists a $x_0(=x)\in A_0$ such that $\bar{x}=\bar{x}_0$.

3. Topologically Cyclic Representation

Let A be a $*$-algebra over the complex field C and X a vector space over C. A $*$-homomorphism $A\to L(X)$ is called a $*$-representation of A on X, where $L(X)$ is an algebra of all linear transformations of X into itself.

LEMMA 3.1. Let A be a locally convex $*$-algebra and let F be an admissible positive functional on A. If a, $b \in A$, then $\overline{(a+b)_0} = (\bar{a}_0 + \bar{b}_0)$.

THEOREM 3.2. Let F be an admissible positive Hermitian functional on the commutative locally convex $*$-algebra A. Then there exists a representation $a \to T_a$ of A on a Hilbert space H such that $(T_a)^* = T_{a^*}$ for all $a \in A_0$.

PROOF. Since A is commutative, L_F is a two-sided ideal and hence X_F is an algebra. Let $\bar{x} = x + L_F$ and define a scalar product in X_F by $(\bar{x}, \bar{y}) = F(y^*x)$, for $x, y \in A$. The completion of X_F with respect to the inner product will be called H, and then H is a Hilbert space.

Let $\bar{x}_0$ be a fixed element of X_F. Since F is admissible, we may assume that $x_0 \in A_0$. Let $\bar{z} \in H$ and assume that $\bar{z}_n \to \bar{z}$ with $\bar{z}_n \in X_F$.
Then

$$||\bar{x}_0\bar{z}_n - \bar{x}_0\bar{z}_m||^2 = (\bar{x}_0\bar{z}_n - \bar{x}_0\bar{z}_m,\ \bar{x}_0\bar{z}_n - \bar{x}_0\bar{z}_m)$$
$$= F((x_0z_n - x_0z_m)^*(x_0z_n - x_0z_m))$$
$$= F((z_n - z_m)^*x_0^*x_0(z_n - z_m))$$

and

$$||\bar{z}_n - \bar{z}_m||^2 = F((z_n - z_m)^*(z_n - z_m)).$$

Since F is admissible,

$$||\bar{x}_0\bar{z}_n - \bar{x}_0\bar{z}_m||^2 \leq M||z_n - z_m||^2 \text{ with } M > 0.$$

Thus $\{\bar{x}_0\bar{z}_n\}$ is a Cauchy sequence with respect to the inner product norm, and hence the sequence converges to an element $\bar{y}$ of H. Similarly we can show that if $\bar{w}_n \to \bar{z}$ with respect to the inner product norm, then $\{\bar{x}_0\bar{w}_n\}$ converges to $\bar{y}$. Now we define the mapping $a\to T_a$ of A on H by

$$T_a\bar{x}=\bar{a}_0\bar{x},\ \ \bar{x}\in H \text{ where } \bar{a}_0=\bar{a}.$$

Then, if $a,\ b\in A$,

$$\begin{aligned} T_{ab}\bar{x}&=(ab)^{-}{}_0\bar{x}=(ab)^{-}\bar{x}=\bar{a}b\bar{x}=\bar{a}_0\bar{b}_0\bar{x}\\ &=(a_0(b_0x))^{-}=T_a(b_0x)^{-}\\ &=T_aT_b\bar{x} \quad \text{for all } \bar{x}\in H. \end{aligned}$$

Similarly $T_{a+b}=T_a+T_b$ and $T_{\lambda a}=\lambda T_a$ for all $\lambda\in C$. Thus $a\to T_a$ defines a representation of A on H.

Consider the restriction of the representation to A_0. Let $a\in A_0$. Since F is admissible, we have

$$\begin{aligned} ||T_a(\bar{x})||^2&=||\bar{a}\bar{x}||^2=(\bar{a}\bar{x},\bar{a}\bar{x})\\ &=F(x^*a^*ax)\\ &\leq M||\bar{x}||^2 \text{ for some } M>0,\ \bar{x}\in X_F. \end{aligned}$$

Hence T_a is a continuous mapping on X_F. Since X_F is dense in H, T_a can be uniquely extended to a continuous mapping $\hat{T}_a$ on H. However if $\bar{x}\in H-X_F$, let $\{\bar{x}_n\}$ be a subset of X_F such that $\bar{x}_n\to x$. Then

$$\begin{aligned} \hat{T}_a(\bar{x})&=\lim\ \hat{T}_a(\overline{x_n})=\lim\ T_a(\overline{x_n})=\lim\ \bar{a}\overline{x_n}\\ &=\bar{a}\bar{x}=T_a(\bar{x}). \end{aligned}$$

Thus $\hat{T}_a=T_a$ and T_a is a continuous function on H for $a\in A_0$. Since T_a is continuous, we can show that $(T_a)^*=T_{a^*}$ by proving that $(T_a)^*(x)=T_{a^*}(x)$ for all $x\in X_F$.

Let $\bar{x}$ and $\bar{y}$ be elements of X_F, then

$$(T_a\bar{x}, \bar{y}) = F(y^*ax) = F((y^*a)x)$$
$$= (x, (\bar{a}^*)\bar{y}) = (x, T_{a^*}\bar{y}).$$

Thus for $a \in A_0$, we have $(T_a)^* = T_{a^*}$.

COROLLARY 3.3. If A_0 is also an algebra e.g., the product of bounded sets of A is bounded, then the restriction of the above representation to A_0 is a $*$-representation of A_0 on H.

Let X be a vector space over C and let K be a subalgebra of $L(X)$. Let z be a fixed vector in X and let $X_z = \{T(z): T \in K\}$. Then X_z is an invariant subspace of X with respect to K. If there exists an element z of a normed space X such that $X_z = X$, then K is said to be *topologically cyclic* and the vector z is called a *topologically cyclic vector.* A representation $x \to T_x$ of A on X is said to be *topologically cyclic* if, when $K = \{T_x: x \in A\}$, there is a vector z in X such that $X_z = X$.

With these definitions we state the following corollary to Theorem 3.2.

COROLLARY 3.4. Let A be a commutative locally convex $*$-algebra with identity. Let F be an admissible positive Hermitian functional on A. Then the representation obtained above is topological cyclic with a cyclic vector h_0 such that $F(x) = (T_x h_0, h_0)$, $x \in A$.

PROOF. Let $h_0 = \bar{1} = 1 + X_F$. Then by definition $T_x h_0 = \bar{x}_0$, so that the set $\{T_x h_0 : x \in A\} = X_F$ and hence is dense in H. Thus h_0 is a topologically cyclic vector. Now let $x \in A$, then there exists $x_0 \in A_0$ such that $\bar{x} = \bar{x}_0$. Thus

$$F(1^*(x-x_0))=F(x-x_0)=F(x)-F(x_0).$$

By the way, $F(1^*(x-x_0))=((x-x_0)^-, \bar{1})$

$$=(\bar{x},\bar{1})-(\bar{x},\bar{1})=0.$$

Consequently $F(x)=F(x_0)$. Therefore $(T_x h_0, h_0)=(\bar{x}_0 h_0, h_0)$ $=(\bar{x}_0\bar{1}, 1)=F(x_0)=F(x)$ for all $x\in A$.

4. Representable Functional

Let F be a linear functional on the locally convex $*$-algebra A and let $a \to T_a$ be a representation of A on a Hilbert space H such that the restriction of the representation to A_0 is a $*$-representation of A_0 on H. Then F is said to be *representable by* $a \to T_a$ provided there exists a topologically cyclic vector $h_0\in H$ such that

$$F(a)=(T_a h_0, h_0) \text{ for all } a\in A.$$

Let $a \to T_a$ be a representation of A on H and let

$$M=\{h\in H : T_a h=0 \text{ for all } a\in A\}.$$

If $M=\{0\}$, we say that the representation is *essential.*

LEMMA 4.1. If the representation $a \to T_a$ is essential, then each of the subspaces $H_h=\{T_a h: a\in A\}$ is cyclic with h as a cyclic vector.

PROOF. [5, p. 206].

THEOREM 4.2. Let F be a Hermitian functional on the pseudo-complete commutative locally convex $*$-algebra A. Then in order for F to be representable, it is sufficient that

(1) for each $x \in A$, there is a $x_0 \in A_0$ such that $\bar{x} = \bar{x}_0$,

(2) $|F(x)|^2 \leq \mu F(x^*x)$, $x \in A$,

where μ is a positive real constant independent of x.

Proof. Assume that F satisfies the conditions and denote by A_1 the pseudo-complete locally convex $*$-algebra obtained by adjoining the identity element to A. Extend the functional F to A_1 by the definition,

$$F(x+\alpha) = F(x) + \mu\alpha \text{ for } x \in A \text{ and } \alpha \text{ a scalar.}$$

Then

$$\begin{aligned} F((x+\alpha)^*(x+\alpha)) &= F((x^*+\bar{\alpha})(x+\alpha)) \\ &= F(x^*x + x^*\alpha + \bar{\alpha}x + \bar{\alpha}\alpha) \\ &\geq F(x^*x) - 2|\alpha||F(x)| + \mu|\alpha|^2 \\ &\geq F(x^*x) - 2|\alpha|\mu^{\frac{1}{2}} F(x^*x)^{\frac{1}{2}} + \mu|\alpha|^2 \\ &= (F(x^*x) - |\alpha|\mu^{\frac{1}{2}})^2. \end{aligned}$$

Thus F is a positive linear functional on A_1 and Theorem 2.6 guarantees that the first condition of admissibility is satisfied on A_1. To show that the second condition is satisfied, let $x+\alpha \in A_1$. Then by hypothesis there exists $x_0 \in A_0$ such that $\bar{x}_0 = \bar{x}$. Consider $x_0 + \alpha$. Then since $\bar{x}_0 = \bar{x}$ and $(x - x_0) \in L_F$,

$$\begin{aligned} &|F[(y+\beta)^*((x_0+\alpha)-(x+\alpha))]|^2 \\ &= |F[(y+\beta)^*(x_0-x)]|^2 \\ &= |F(y^*(x-x_0)) + F(\bar{\beta}(x_0-x))|^2 \\ &= |\bar{\beta}F(x_0-x)|^2 \\ &\leq |\beta|^2 F[(x_0-x)^*(x_0-x)] = 0. \end{aligned}$$

Consequently $(x_0+\alpha)^{-} = (x+\alpha)_0^{-}$.

Therefore F is an admissible positive Hermitian functional on A_1. Hence by Corollary 3.4 there exists a representation $x \to T_x$ of A_1 on H defined by $T_{(a+\alpha)}x = (a+\alpha)_0\bar{x}$ and such that

$$F(a+\alpha) = (T_{a+\alpha}h_0, h_0) \text{ for some } h_0 \in H.$$

Now let $N = \{h \in H : T_a h = \theta \text{ for all } a \in A\}$.

Consider the restriction of $a \to T_a$ to the space $N^\perp$, where

$$N^\perp = \{h \in H : (h, n) = 0 \text{ for all } n \in N\}.$$

Since $\{h \in N^\perp : T_a h = \theta \text{ for all } a \in A\} = \{0\}$, the restriction is essential.

Let $h_0 = h_0' + h_0''$ where $h_0' \in N^\perp$ and $h_0'' \in N$. Then for all $a \in A$ we have

$$\begin{aligned} F(a) &= (T_a h_0, h_0) = (T_a(h_0' + h_0''), \ h_0' + h_0'') \\ &= (T_a h_0', \ h_0' + h_0'') = (h_0', \ T_a^*(h_0' + h_0'')) \\ &= (h_0', \ T_a^* h_0') = (T_a h_0', h_0'). \end{aligned}$$

Thus there exists $h_0' \in N^\perp$ such that $F(a) = (T_a h_0', h_0')$ for all $a \in A$. Let $H_0 = \{T_a h_0' : a \in A\}$. Then, since the restriction of the representation to $N^\perp$ is essential, by Lemma 4.1 H_0 is cyclic with h_0 as a cyclic vector.

COROLLARY 4.3. If A has an identity element, then every positive functional which implies condition (1) is representable.

PROOF. If A has an identity element, then by the Cauchy-Schwarz inequality, we have

$$|F(x)|^2 \leq F(1)F(x^*x)$$

for any positive functional F. Thus, condition (2) is automatically satisfied.

COROLLARY 4.4. Let F be an admissible positive Hermitian functional on the pseudo-complete commutative locally convex $*$-algebra A. Then there exists a $*$-representation of A_0 on a Hilbert space H.

PROOF. If A is commutative and pseudo-complete, then *A_0 is an subalgebra of A [1].* Therefore by Theorem 3.2 and Corollary 3.3, the proof is obvious.

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Department of Mathematics
Ch'angwon National University
Ch'angwon 615
Korea

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