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SOME PROPERTIES OF PROJECTIVE REPRESENTATIONS OF SOME FINITE GROUPS

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The representation group G^* of metacyclic group $G=BH$, $H \Delta G$ is known. When $|B|$ is prime, representation G^* can be easily obtained. Using this fact, some properties of projective representation of G will be discussed.

THEOREM 1. Let $G=\langle x,y|x^n=1,\ y^p=1,\ y^{-1}xy=x^r\rangle$ where $(n,r)=1$, p is prime. Then the number of irreducible projective representation with degree 1 is $p(n,r-1)$ and the degree of the irreducible projectve representation of G is one or p.

PROOF. $H^2(G, K^*)\cong Z_q$ where $q=\dfrac{k(n,r-1)}{n}$, $k=\left(n, \dfrac{r^p-1}{r-1}\right)$. By [3] the representation group G^* of G is

$$\langle x,y,z|x^n=1,\ y^p=1,\ z^q=1,\ y^{-1}xy=zx^r,\ xz=zx,\ yz=zy\rangle.$$

Also $\langle x,z\rangle \triangleleft G^*$, $\langle y\rangle < G^*$ and $\langle x,z\rangle$ is abelian. So G^* is the semidirect product of $\langle y\rangle$ by $\langle x,z\rangle$.

Let T be a representation of $\langle x,z\rangle$. We define $T^a: h\longrightarrow T(a^{-1}a^h)$ for $h\in\langle x,z\rangle$ and $a\in\langle y\rangle$. Then $S_T=\{y^k\in\langle y\rangle | Ty^k \cong T\}$ is a subgroup of $\langle y\rangle$. Since $|\langle y\rangle|$ is prime, $S_T=\{e\}$

or $S_T=\langle y\rangle$. So by [4], $S_T=\langle y\rangle$ iff $(T\otimes\rho)$ has degree 1, where ρ is an irreducible representation of $\langle y\rangle$.

All the irreducible representation whose degree is 1 has the form $T\otimes\rho$. So we have

$$\begin{aligned} Ty^k\cong T \text{ iff } & Ty^k(x^iz^j)=T(x^iz^j) \\ \text{iff } & Ty^k(x^iz^j)=T(y^{-k}x^iz^jy^k) \\ & \qquad =T(x^{ir^k}z^{j+(1+r+\cdots+r^{k-1})}) \\ & \qquad =T(x)^{ir^k}T(z)^{j+(1+r+\cdots+r^{k-1})} \\ & \qquad =T(x)^iT(z)^j \\ \text{iff } & T(x)^{i(r^{k-1})}=1 \text{ and } T(z)^{1+r+\cdots+r^{k-1}}=1 \\ \text{iff } & d_1\frac{1-r^k}{1-r}\equiv 0(\text{mod } n) \text{ and } d_2\ \frac{1-r^k}{1-r}\equiv 0(\text{mod } q) \end{aligned}$$

where $T(x)=\xi_1{}^{d_1}$, $T(z)=\xi_2{}^{d_2}$.

(ξ_1,ξ_2 are n, q-th roots of 1, respectively.)

So

$$\begin{aligned} S_T=\langle y\rangle \text{ iff } & d_1(1-r^k)\equiv 0(\text{mod } n) \text{ and} \\ & d_2\frac{1-r^k}{1-r}\equiv 0(\text{mod } q) \end{aligned}$$

for all k, $0\leq k\leq p-1$

and

$$S_T=\langle y\rangle \text{ iff } d_1(1-r)\equiv 0(\text{mod } n),$$

because $(1+r,\ 1+r+r^2,\cdots,1+r+\cdots+r^{k-1})=1$. So such $\{d_1\}$ is $(n,r-1)$. Therefore $\{T\otimes\rho\}$ is $p(n,\ r-1)$.

Since S_T is $\{e\}$ or $\langle y\rangle$, the degree of irreducible representation G^* is 1 or p. So the degree of projective irreducible representation of G is 1 or p. So our proof is completed.

THEOREM 2. Let $G=\langle x,\ y|x^n=1=y^p,\ y^{-1}xy=x^r\rangle$ with $(r,n)=1$, p prime, and $1+r+\cdots+r^{p-1}\equiv 0 (\bmod n)$.

(1) Then $p=q$, $p|n$ and $H^2(G,\ K^*)=\{1,\{\alpha\},\cdots,\{\alpha^{p-1}\}\}$.

(2) For each $\{\alpha^k\}$, there exists exactly n/p linearly inequivalent projective representations with factor set $\{\alpha^k\}$.

(3) In this case

$$T_{ki}(x)=\mathrm{diag}\{\xi^{k+i},\ \xi^{k(1+r)+ri},\cdots,\xi^{k(1+r+\cdots+r^{p-1})+r^{p-1}i}\},$$

$$T_{ki}(y)=\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

$$T_{ki}(y^j x^l)=T_{ki}(y)^j T_{ki}(x)^l,$$

and

$$\alpha(y^j x^i,\ y^l x^m)=\xi^{(1+r+\cdots+r^{l-1})i},$$

where ξ is a primitive n-th root of unity.

PROOF. In this situation we have $k=\left(n,\ \dfrac{r^p-1}{r-1}\right)=n$, $q=k(n,r-1)/n=(n,\ r-1)=d$. Therefore $r^p-1=(r-1)(r^{p-1}+\cdots+1)\equiv 0(\bmod n)$ and hence $0=1+r+\cdots r^{p-1}=1+1+\cdots+1\equiv p(\bmod d)$. But p is prime so $d=1$ or $d=p$. So $d=q=1$, $H^2(G,C^*)=\{e\}$, $d=p=q$, and $p|n$ since $(n,\ r-1)=d=p=q$. Now by [1], T_{ki} is a projective irreducible representation of G wich degree p with the factor set $\{\alpha^k\}$. Also their equivalence can be found in [1]. So for each $\{\alpha^k\}$ we have n/p linearly inequivalent projective representation.

References

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