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GALOIS SUBRINGS OF UTUMI RINGS OF QUOTIENTS

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1. Introduction

For the Galois theory of semiprime rings the importance of crossed products was firstly noticed by T. Nakayama in [8]. ~~After his proving the beautiful normal basis~~ theorem crossed product has greatly influenced to Galois theory. But still at his time there was no systematic scheme between crossed products and Galois subrings. M. Cohen [3] successfully provided a systematic "Morita context" between them by the hint of Chase, Harrison and Rosenberg. But actually J. Osterburg and J. Park pointed out that her context is the derived context of some of module.

As J. Osterburg and J. Park did in [9] we consider crossed products and Galois subrings altogether at the same time via the drived Morita context. Indeed we prove that the Utumi quotient ring of Galois subring is the Galois subring of the Utumi quotient ring in different way. Also we consider the normal basis theorem for regular self-injective ring case. By our normal basis theorem we can generalize M. Cohen's result [3] for the semi-simple artinian ring case. Some our results in this paper were alrealy proved in [9].

2. Preliminaries

In this section some necessary definitions and properties will be given. All rings are assumed to have identity element. R denotes a ring and all modules are right R-modules, unless otherwise mentioned.

DEFINITION 2.1. A *Morita context* consists of two rings R and S, two bimodules ${}_SP_R$ and ${}_RQ_S$, and two bimodule homomorphisms (called the pairings)

$$(\ , \ ):\ Q\otimes_S P \longrightarrow R$$

and

$$[\ , \ ]:\ P\otimes_R Q \longrightarrow S$$

satisfying the associativity conditions $q[p, q']=(q, p)q'$ and $p(q, p')=[p, q]p'$.

The images of the pairings are called the *trace ideals* of the context, and are denoted by T_R and T_S. We abbreviate a context by the symbol $\langle P, Q\rangle$.

For any R-module P_R let $P^*=Hom(P_R, R_R)$ and $S=End(P_R)$. Then P^* is a right S and left R-bimodule. Define pairings $(\ , \ )$ and $[\ , \ ]$ as follows; $(\ , \ ):\ P^*\otimes_S P\longrightarrow R$ by $(f, p)=f(p)$ and $[\ , \ ]:\ P\otimes_R P^*\longrightarrow S$ by $[p, f](x)=pf(x)$ for x in P. Then it can be easily checked that (R, P, P^*, S) is a Morita context between R and S. This particular context is called the *derived Morita context* of P_R. For the left module case, we can define the derived Morita context similarly.

EXAMPLE 2.2. From two module X_A and T_A, define

$$R=End_A(X),\ S=End_A(Y),$$
$$P=Hom_A(X, Y),\ \text{and}\ Q=Hom_A(Y, X)$$

with pairings by composition. Then $\langle P, Q\rangle$ is the Morita context of two rings R and S.

EXAMPLE 2.3. Let $\langle A, B, P, Q, \alpha, \beta\rangle$ be a Morita context. Define the generalized matrix ring

$$R=\begin{bmatrix} A & P \\ Q & B \end{bmatrix},$$

α, β by using ordinary matrix addition and multiplication by means of α and β. Then R is actually a ring.

Put
$$e=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then
$$A=eRe, \qquad B=(1-e)R(1-e),$$
$$P=eR(1-e), \qquad Q=(1-e)Re.$$

In general R is an arbitary ring with an idempotent e, then $\langle eRe, (1-e)R(1-e), eR(1-e), (1-e)Re, \alpha, \beta\rangle$ is a Morita context for suitable α, β.

Associated with any Morita context $\langle P, Q\rangle$ there are eight natural maps, e.g., $p\in P\to[p, -]\in Q^*=Hom_S(Q, End_R(P))$ and $r\in R\to(q\to rq)\in End_S(Q)$, where $Q=Hom_R(P, R)$ and $S=End_R(P)$.

DEFINITION 2.4. The context $\langle P, Q\rangle$ is called *non-degenerate* if all these natural maps are injective.

DEFINITION 2.5. A Morita context is *right normalized* if the four natural maps $P\to Q^*$, $Q\to P^*$, $R=End_S(Q)$ and $S\to End_R(P)$ are isomorphisms.

THEOREM 2.6 [7, Theorem 19]. If a Morita context $\langle P, Q\rangle$ between two rings R and S is nondegenerated, then the maximal quotient Morita context $\langle P, Q\rangle$ between $Q_{max}(R)$

and $Q_{max}(S)$ induced by $\langle P, Q\rangle$ is right normalized.

Let A be a finite dimensional central simple algebra over its center field F. Then A has a maximal subfield K. If K is a Galois extension of F with a Galois group G then there are invertible elements $\{a_g: g \text{ in } G\}$ of A such that $A=\oplus Ka_g$, a direct sum of left K-vector spaces, where for all x in K, $xa_g=a_g g(x)$. Moreover defining $t(g, h)=a_g a_h a_{gh}^{-1}$ for each g, h in G, we have $t(g, h)$ is in K and the following equations hold for all g, h, k in G:

$$t(g, h)t(gh, k)=t(h, k)t(g, k).$$

By this fact we define a crossed product formally as ~~follows~~.

DEFINITION 2.7. Let R be a ring and G be a group. Given a group homomorphism $p: G\rightarrow \mathrm{Aut}(R)$ and a map $t: G\times G\rightarrow U(R)$ the units of R such that

$$(1) \qquad t(x, y)t(xy, z)=t(y, z)^{p(x)}t(x, yz)$$

and

$$(2) \qquad t(x, y)a^{p(xy)}=a^{p(x)p(y)}t(x, y)$$

for all x, y, z in G and a in R. We define the *crossed product* $R*G$ to be the set of all formal sums of the form $\sum a_x\bar{x}$ with a_x in R and $a_x=0$ for almost all x in G. The addition in $R*G$ is defined componentwise and the multiplication is given by the rule

$$(a_x\bar{x})(a_y\bar{y})=a_x a_y^{p(x)}t(x, y)\overline{xy}.$$

This makes $R*G$ an associative ring with identity $t(1, 1)^{-1}\bar{1}$. When $t(x, y)=1$ for every x, y in G, the crossed product is called a *skew group ring* and denoted by RG.

EXAMPLE 2.8. Let G be a finite group of automorphisms of a field F. Then the skew group ring FG is the $n\times n$ matrix ring over the fixed field F^G, where n is the order of G.

We set some notations and basic definitions. Let G be a group of automorphisms acting on R. By r^g we mean the image of r under g in G. The fixed ring of R is $R^G=\{r$ in $R \mid r^g=r$ for all g in $G\}$. The *trace* of x is $tr(x)=\sum x^g$. Note that $tr(x)$ is in R^G. An ideal L(left or two sided) of R is called G-invariant if L^g is contained in L for all g in G. The ring R is said to have no $|G|$-torsion if $|G|r=0$ for r in R implies that $r=0$. If $\mathscr{F}$ is a Gabriel filter on R, then $t(R_R)$ is the set of all a in R whose right annihilator is a member of $\mathscr{F}$ and is called the torsion submodule of R_R with respect to $\mathscr{F}$.

3. Main Results

When G is a finite group of automorphisms of R we can form a skew group ring $S=RG$ over R.

For a given Gabriel filter $\mathscr{F}$ on R, $\overline{\mathscr{F}}=\{\bar{L}\triangleleft_r RG \mid \bar{D}\cap R$ is in $\mathscr{F}\}$ is a Gabriel filter on RG by K. Louden [6, Lemma 8].

We recall that $t(R_R)$ is the torsion submodule of R_R with respect to $\mathscr{F}$, $t(S_R)$ is the torsion submodule of S_R with respect to $\mathscr{F}$ and $t(S_S)$ is the torsion submodule of S_S with respect to $\overline{\mathscr{F}}$. Then we can obtain $t(R_R)S=t(S_S)$ and $t(S_R)=t(S_S)$. Furthermore $t(R_R)=t(S_R)\cap R$. We call that $\mathscr{F}$ is a G-invariant (or an automorphism invariant) if I^g is an element of $\mathcal{F}$ for all I in $\mathcal{F}$, g in G.

LEMMA 3.1. If $\mathcal{F}$ is an automorphism invariant, then every automorphism of R can be extended to an automorphism of $Q(R)$.

PROOF. Let $t(R)$ be the torsion submodule of R with respect to $\mathcal{F}$. Since $\mathcal{F}$ is an automorphism invariant, $g(t(R))=t(R)$ for all g in Aut (R). Let f be an R-homomorphism from D to $R/t(R)$ which represents an element of $Q(R)$. Define $g(f)$ from $g(D)$ to $R/t(R)$ by $g(f)(g(d))=g(f(d))$ for d in D. Then $g(f(R))=t(R)g(f)$ is well-defined. For, if $f(d_1)=f(d_2)$ for $f(d_1)$, $f(d_2)$ in $R/t(R)$. Put $f(d_1)=r_1+t(R)$ and $f(d_2)=r_2+t(R)$ for r_1, r_2 in R. Since $r_1+t(R)=r_2+t(R)$ we have r_1-r_2 is in $t(R)$ and hence $g(r_1-r_2)$ is in $t(R)$. It follows that $g(f(d_1))=g(f(d_2))$. Therefore $g(f)(g(d_1))=g(f)(g(d_2))$ which completes the well-definedness. Next we show that $g(f)$ is a homomorphism. Since $g(f)(g(d_1)+g(d_2))=g(f)(g(d_1+d_2))=g(f(d_1+d_2))=g(f(d_1)+f(d_2))=g(f(d_1))+g(f(d_2))=g(f)g(d_1)+g(f)g(d_2)$ and $g(f)(g(d_1)r)=g(f)(g(d_1r))=g(f(d_1r))=g(f(d_1)r)=g(f(d_1))r=g(f)(g(d_1))r$ for all $g(d_1), g(d_2)$ in $g(D)$ and r in R. Thus $g(f)$ is a homomorphism and this defines g on $Q(R)$ to be an automorphism.

By Lemma 3.1, if $\mathcal{F}$ is a G-invariant filter, then G can be considered as an automorphism group on $Q_{\mathcal{F}}(R)$. Let $[q]$ be in $Q_{\mathcal{F}}(R)$ which is represented by $q: D\to R/t(R)$ with D in $\mathcal{F}$. Define $\bar{q}: DS\to S/t(S)$ by $\bar{q}(\sum d_g g)=\sum q(d_g)g$. Then since $R/t(R)$ is contained in $S/t(S_R)$ and $t(S_R)=t(S_S)$, $\bar{q}$ is well-defined and an S-homomorphism and $\bar{q}|D=q$ with

DS in $\bar{\mathscr{F}}$. Let $[\bar{q}]$ be in $Q_{\bar{\mathscr{F}}}(S)$ represented by this map $\bar{q}$. For any g in G, the left multiplication $L_g: S_S \longrightarrow S_S$ induces an S-homomorphism $\bar{L}_g: S/t(S_S) \longrightarrow S/t(S_S)$. So $\bar{L}_g\bar{q}$: $DS \longrightarrow S/t(S_S)$ represents an element $[\bar{L}_g\bar{q}]$ in $Q_{\bar{\mathscr{F}}}(S)$.

THEOREM 3.2. Let G be a finite group of automorphisms of R and $S=RG$. Then for a G-invariant filter $\mathscr{F}$ on R, $Q_{\mathscr{F}}(R)G$ is isomorphic to $Q_{\bar{\mathscr{F}}}(S)$.

PROOF. Define f: $Q_{\mathscr{F}}(R)G \longrightarrow Q_{\bar{\mathscr{F}}}(S)$ by $f(\Sigma g[q_g]) = \Sigma[\bar{L}_g\bar{q}_g]$. We divide the proof into five steps.

STEP 1. We show that f is well-defined.

If $\Sigma g[q_g] = \Sigma g[w_g]$, then $[q_g] = [w_g]$ for all g in G. Thus there exists D_g in $\mathscr{F}$ such that q_g and w_g are coincided on D_g for all g in G. Therefore for all g in G, $\bar{q}_g$ and $\bar{w}_g$ agree on D_gS and D_gS is in $\bar{\mathscr{F}}$. So for all g in G, $\bar{L}_g\bar{q}_g = \bar{L}_g\bar{w}_g$ on D_gS. Hence $[\bar{L}_g\bar{q}_g] = [\bar{L}_g\bar{w}_g]$ for all g in G. Consequently $\Sigma[\bar{L}_g\bar{q}_g] = \Sigma[\bar{L}_g\bar{w}_g]$. Hence $f(\Sigma g[q_g]) = f(\Sigma g[w_g])$ and therefore f is well-defined.

STEP 2. We show that f is additive.

Since $[L_g][\bar{q}_g] = [L_g\bar{q}_g]$ and $[\overline{q_g + w_g}] = [\bar{q}_g] + [\bar{w}_g]$, we have following:

$$\begin{aligned} & f(\Sigma g[q_g] + g[w_g]) = f(\Sigma g([q_g] + [w_g])) \\ &= f(\Sigma g[q_g + w_g]) = \Sigma[\bar{L}_g(\overline{q_g + w_g})] = \Sigma[\bar{L}_g][\overline{q_g + w_g}] \\ &= \Sigma[\bar{L}_g]([\bar{q}_g] + [\bar{w}_g]) = \Sigma[\bar{L}_g][\bar{q}_g] + \Sigma[\bar{L}_g][\bar{w}_g] \\ &= \Sigma[\bar{L}_g\bar{q}_g] + \Sigma[\bar{L}_g\bar{w}_g] = f(\Sigma g[q_g]) + f(\Sigma g[w_g]). \end{aligned}$$

Hence f is additive.

STEP 3. We show that f is a homomorphism.

Choose $g[q_g]$ and $h[w_h]$ in $Q_{\mathscr{F}}(R)G$ with $q_g:D_1\longrightarrow R/t(R)$ and w_h: $D_2\longrightarrow R/t(R)$ for g, h in G. Then we have $g[q_g]h[w_h]=gh[q_g]^h[w_h]$. Let $[x]=[q_g]^h$. Then $[x]$ is represented by $x:D_1{}^h\longrightarrow R/t(R)$; $x(d_1{}^h)=q_g(d_1)^h$ for d_1 in D_1. So $g[q_g]h[w_h]=gh[x][w_h]$. Let $y=[xw_h]$. Then $[y]$ is represented by y; $w_h{}^{-1}(D_1{}^h/t(D_1{}^h))\xrightarrow{w_h}D_1{}^h/t(D_1{}^h)\xrightarrow{\bar{x}}R/t\,(R)$, where $\bar{x}$ is the induced R-homomorphism from x. Hence $f\,(g[q_g]h[w_h])=f\,(gh[y])=[\bar{L}_{gh}\bar{y}]$ is represented by S-homomorphism: $w_h{}^{-1}(D_1{}^h/t(D_1{}^h))S\xrightarrow{\bar{y}}S/t(S)\xrightarrow{\bar{L}_{gh}}S/t(S)$.

On the other hand, $f(g[q_g])f\,(h[w_h])=[\bar{L}_g\bar{q}_g][\bar{L}_h\bar{w}_h]$ is represented by the composition; $\bar{w}_h{}^{-1}\bar{h}^{-1}(D_1S/t(D_1S))\xrightarrow{\bar{w}_h}$ $\bar{L}_h{}^{-1}(D_1S/t(D_1S))\xrightarrow{\bar{L}_h}D_1S/t(D_1S)\xrightarrow{\bar{\bar{q}}_g}S/t(S)\xrightarrow{\bar{L}_g}S/t(S)$ where $\bar{\bar{q}}_g$ is the induced map from $\bar{q}_g$. In this case $\bar{h}^{-1}(D_1S/t(D_1S))=D_1{}^hS/t(D_1{}^hS)$ and $\bar{w}_h{}^{-1}\bar{L}_h{}^{-1}(D_1S/t(D_1S))=w_h{}^{-1}(D_1{}^hS/t(D_1{}^hS))=w_h{}^{-1}(D_1{}^h/t(D_1{}^h))S$. Let $D_3=w_h{}^{-1}(D_1{}^h/t(D_1{}^h))$. Then for d in D_3 and k in G, $(\bar{L}_{gh}\bar{y})(dk)=\bar{L}_{gh}(y(d)k)=\bar{L}_{gh}y(d)k=\bar{L}_{gh}[\bar{x}w_h(d)]k$. Let $w_h(d)=d_1{}^h+t(D_1{}^h)$ with d_1 in D_1. Then $\bar{L}_{gh}[\bar{x}w_h(d)]k=\bar{L}_{gh}[\bar{x}(d_1{}^h)+t(D_1{}^h)]k=\bar{L}_{gh}[x(d_1{}^h)]k=\bar{L}_{gh}[q_g(d_1)^h]k=\bar{L}_g[q_g(d_1)]hk$. And we have $(\bar{L}_g\bar{\bar{q}}_g\bar{L}_hw_h)(dk)=(\bar{L}_g\bar{\bar{q}}_g\bar{L}_h)\,(w_h(d)k)=(\bar{L}_g\bar{\bar{q}}_g\bar{L}_h)\,(d_1{}^h+t(D_1{}^h))k=(\bar{L}_g\bar{\bar{q}}_g)(hd_1{}^hk+t(S))=(\bar{L}_g\bar{\bar{q}}_g)\,(d_1hk+t(S))=\bar{L}_g(q_g(d_1)hk)=\bar{L}_g\,[q_g(d_1)]hk$. Hence $f(g[q_g]h[w_h])=f\,(g[q_g])f(h[w_h])$. Therefore f is a homomorphism.

STEP 4. We show that f is one to one.

Suppose $[\sum\bar{L}_g\bar{q}_g]=0$ with $\bar{q}_g:D_gS\longrightarrow S$ with D_g in $\mathscr{F}$, g in G. Then $\sum[\bar{L}_g\bar{q}_g]$ is represented by the S-homomorphism: $\bigcap(D_gS)\longrightarrow S/t(S)$; $x\longrightarrow\sum\bar{L}_g(\bar{q}_g(x))$. Since $\sum[\bar{L}_g\bar{q}_g]$

$=0$, there exists a G-invariant D_0 in $\mathscr{F}$ such that D_0S is contained in $\bigcap(D_gS)$ and $\bar{L}_g(q_g(x))=0$ on D_0S. Now for every d_0 in D_0, $0=\bar{L}_g(\bar{q}_g(d_0)) = \sum\bar{L}_g(q_g(d_0))$. Let $q_g(d_0)=r_g+t(R_R)$ with r_g in R. Then $0=\sum\bar{L}_g(q_g(d_0))=\sum\bar{L}_g(r_g+t(R))=\sum \bar{L}_g(r_g+t(S)) = \sum (gr_g+t(S))=(\sum r_g^{g^{-1}}g)+t(S)$. Hence $\sum r_g^{g^{-1}}g$ is a element of $t(S)=t(R)S$. So for all g in G, $r_g^{g^{-1}}$ is contained in $t(R)$. Hence r_g is a element of $t(R)$ for all g in G. Therefore $q_g(d_0)=r_g+t(R)=0$ for all g in G and d_0 in D_0. Hence $q_g=0$ on D_0.

Therefore $[q_g]=0$ and so $\sum g[q_g]=0$. Hence Ker $f=\{0\}$. Thus f is $1-1$.

STEP 5. We show that f is onto.

Let $[x]\in Q_{\mathscr{F}}(S)$ represented by $x: DS\longrightarrow S/t(S)$ with D in $\mathscr{F}$. Let p be a map from $S/t(S)$ to $R/t(R)$ defined by $p(\sum r_h h+t(S))=r_g{}^g+t(R)$ for all g in G. Since $t(R_R)S=t(S_S)$, p is well-defined and an R-homomorphism.

$$\begin{tikzcd} D_R \arrow[r, "x"] \arrow[dr, "px"'] & S/t(S) \arrow[d, "p"] \\ & R/t(R) \end{tikzcd} \qquad \begin{tikzcd} d \arrow[r, mapsto] \arrow[dr] & r(d)_h^h + t(S) \arrow[d] \\ & r(d)_g^g + t(R_R) \end{tikzcd}$$

where $r(d)_h$ is a element of R for all h in G. Thus px is an R-homomorphism: $D_R\longrightarrow R/t(R)$. We will show that $[x]=\sum[\bar{L}_g\,\bar{p}\bar{x}]=f(\sum g[px])$. $\sum[\bar{L}_g\,\bar{p}\bar{x}]$ is represented by $DS\longrightarrow S/t(S)$; $y\longmapsto\sum\bar{L}_g[(px)(y)]$. Now for d in D and h in G; $\sum\bar{L}_g[\bar{p}\bar{x}(dh)]=\sum\bar{L}_g([(px)(d)]h)=\sum(\bar{L}_g[(px)(d)])h=\sum\bar{L}_g[r(d)_g{}^g+t(R)]h=\sum\bar{L}_g(r(d)_g{}^g+t(S))h=\sum(gr(d)_g+t(S))h=\sum(r(d)_g g+t(S))h=\sum(r(d)_g+t(S))gh$ and $x(dh)=$

$x(d)h=\sum(r(d)_g g+t(S))h=\sum(r(d)_g+t(S))gh$. Hence $[x]=\sum[\bar{L}_g\bar{p}\bar{x}]=f(\sum g[\bar{p}\bar{x}])$. Thus f is onto. Resultly f is an isomorphism. Thus $Q_{\mathscr{F}}(R)G$ is isomorphic to $Q_{\bar{\mathscr{F}}}(S)$.

An overring S of a ring R with same identity is called a *finite normalizing extension ring* of R if S is finitely generated as an R-module by elements which normalize R, that is, $S=\sum_{i=1}^{n} Rx_i$ with $Rx_i=x_iR$ for each i.

For example, it includes crossed product $R*G$ with a finite group G.

LEMMA 3.3 [5, Theorem 3.2]. If $S=\sum_{i=1}^{n} x_iR$ is a finite normalizing extension of R with $X_1=1_R=1_S$, then for M in *Mod-R*, $Hom_R(S_R,M_R)$ is an injective S-module if and only if M is an injective R-module.

Immediately by the above Lemma we can get $Hom_R(S_R, E_R(R))=E_S(Hom_R(S_R, R_R))$.

LEMMA 3.4. Let $S=RG$ be a skew group ring with a finite group G, and let $\mathscr{F}$ be the Lambek topology (or topology) on R. Then $\bar{\mathscr{F}}=\{I\triangleleft_r RG : I\cap R\in\mathscr{F}\}$ is the Lambek topology on RG.

PROOF. By the above Lemma 3.3, we have $Hom_R(RG, E(R)_R)=Hom_{RG}(Hom_R(RG, R_R))=E(RG)_{RG}$. Since $E(R)_R$ is an injective cogenerator, $Hom_R(RG, E(R))=E(RG)_{RG}$ is an injective cogenerator by K. Louden [6, Proposition 4]. Therefore $\bar{\mathscr{F}}$ is also a Gabriel filter on RG.

COROLLARY 3.5. $Q_{\max}(R)G$ is isomorphic to $Q_{\max}(RG)$.

PROOF. Define f: $Q_{\max}(R)G \longrightarrow Q_{\max}(RG)$ by $f(\sum g[q_g]) = \sum [L_g \bar{q}_g]$.

STEP 1. f is well-defined.

It is obvious that $\sum [L_g \bar{q}_g]$ is in $Q_{\max}(S)$. Now if $\sum g[q_g] = \sum g[w_g]$ with $[q_g]$, $[w_g]$ in $Q_{\max}(R)$. Then we can obtain $[q_g] = [w_g]$ for all g in G. Therefore there exists D_g in $\mathscr{F}$ such that $q_g|D_g = w_g|D_g$ for all g in G. Thus we have $\bar{q}_g|D_gS = \bar{w}_g|D_gS$ and D_gS in $\bar{\mathscr{F}}$ by Lemma 3.4 for all g in G. Hence $L_g\bar{q}_g|D_gS = L_g\bar{w}_g|D_gS$ for all g in G. Therefore we have $[L_g\bar{q}_g] = [L_g\bar{w}_g]$ for all g in G. Consequently, $f(\sum g[q_g]) = \sum [L_g\bar{q}_g] = \sum [L_g\bar{w}_g] = f(\sum g[w_g])$. Thus f is well-defined.

STEP 2. f is additive.

Since $[L_g][\bar{q}_g] = [L_g\bar{q}_g]$ and $[\overline{q_g + w_g}] = [\bar{q}_g] + [\bar{w}_g]$, f is clearly additive.

STEP 3. f is a homomorphism.

Suppose $f(g[q_g]) = [L_g\bar{q}_g]$ with q_g: $D_1 \longrightarrow R_R$ and $f(h[w_h]) = [L_h\bar{w}_k]$ with $w_k: D_2 \to R_R$. Then $f(g[q_g]\,h[w_h]) = f(gh[q_g]^h [w_h])$. Let $[u] = [q_g]^h$. Then $[u]$ is represented by u: $D_1{}^h \to R_R$; $u(d_1{}^h) = q_g(d_1)^h$ for d_1 in D_1. So $f(g[q_g]h[w_h]) = f(gh[u][w_h])$. Let $[v] = [u][w_h]$ and let $D_3 = w_h{}^{-1}(D_1{}^h)$. Then $v: D_3 \xrightarrow{w_h} D_1 \xrightarrow{u} R_R$ represents $[v]$. So $f(g[q_g]h[w_h]) = f(L_{gh}[v]) = [L_{gh}\bar{v}]$; $D_3 \xrightarrow{\bar{v}} S_S \xrightarrow{L_{gh}} S_S$. Now for $f(g[q_g])f(h[w_k]) = [L_g\bar{q}_g]\,[L_h\bar{w}_h]$ is represented by the composition; $w_h{}^{-1}L_k{}^{-1}(D_1S) \xrightarrow{\bar{w}_h} L_h{}^{-1}(D_1S) \xrightarrow{L_h} D_1S \xrightarrow{\bar{q}_g} S_S \xrightarrow{g} S_S$. In this case $L_h{}^{-1}(D_1S) = D_1{}^hS$ and $\bar{L}_h{}^{-1}(L_h{}^{-1}(D_1S)) = \bar{L}_h{}^{-1}(d_1{}^hS) = L_h{}^{-1}(D_1{}^h)S = D_3S$. Now for dk in D_3S with d in D_3 and k in G;

$(L_{gh}\bar{v})(dk)=gh(v(d)k)=gh\ u(w_h(d))k=gh\ u[(w_h(d)^{h^{-1}})^h]k$
$=gh[q_g(w_h(d)^{h^{-1}})]^h k=g[q_g(w_h(d)^{h^{-1}})]hk.$

On the other hand $(L_g\bar{q}_gL_h\bar{w}_h)(dk)=(L_g\bar{q}_gL_h)(w_h(d)k)=(L_g\bar{q}_g)(hw_h(d)k)=(L_g\bar{q}_g)(w_h(d)^{h^{-1}}hk)=L_g(\bar{q}_g(w_h(d)^{h^{-1}}hk))$ $=g(q_g(w_h(d)^{h^{-1}})hk)=g(q_g(w_h(d)^{h^{-1}}))hk$. Hence $L_{gh}\bar{v}|D_3S=L_g\bar{q}_gL_h\bar{w}_h|D_3S$. Therefore

$$f(g[q_g]h[w_h])=f(g[q_g])\,f(h[w_h]).$$

So f is a homomorphism.

STEP 4. f is one to one.

Suppose $\sum[L_g\bar{q}_g]=0$ with $\bar{q}_g: D_gS_S \longrightarrow S_S$. Then $\sum[L_g\bar{q}_g]$ is represented by S-homomorphism;

$$\bigcap(D_gS)\longrightarrow S_S.$$
$$x\longrightarrow\sum g(\bar{q}_g(x)).$$

Since $\sum[L_g\bar{q}_g]=0$, there exists G-invariant D_0 in $\mathscr{F}$ and $D_0S\subseteq\bigcap(D_gS)$ such that $\sum g(\bar{q}_g(x))=0$ for all x in D_0S. Now for d_0 in D_0, $0=\sum g(\bar{q}_g(d_0))=\sum q_g(d_0)^{g^{-1}}g$. Hence $q_g(d_0)^{g^{-1}}=0$ and so $q_g(d_0)=0$ for all g in G and d_0 in D_0. So $q_g=0$ on D_0, $q_g=0$ on D_g for all g in G. So $[q_g]=0$. Hence we have $\sum g[q_g]=0$ for all g in G. Therefore $\mathrm{Ker}f=\{0\}$. Thus f is one to one.

STEP 5. f is onto.

$$\begin{tikzcd} D_R \arrow[r, "x"] \arrow[dr, "Dx"'] & S_R \arrow[d, "p"] \\ & R_R \end{tikzcd}$$

Let $[x]$ be element of $Q_{max}(S)$ represented by $x: DS_S\to S_S$ with D in $\mathscr{F}$.

Define p: $S_R\to R_R$ by $p(\sum r_h h)=r_g{}^g$ for all g in G.

Then p is an R-homomorphism. So $x(d)=\sum[(px)(d)]^{g^{-1}}g$ for all d in D. So $[px]$ in $Q_{max}(R)$ represented by px: $D_R \longrightarrow R_R$.

Claim. $[x]=\sum[L_g\bar{p}\bar{x}]$

$\sum[L_g\bar{p}\bar{x}]$ is represented by the S-homomorphism; $DS_S \longrightarrow S_S$: $y \longmapsto \sum g(\bar{p}\bar{x}(y))$. Now for d in D and h in G, $x(dh)=x(d)h=\sum[(px)(d)]^{g^{-1}}gh$ and $\sum g(\bar{p}\bar{x}(dh))=\sum g[(px)(d)h]=(\sum g(px)(d))h=\sum[(px)(d)]^{g^{-1}}gh$. Hence $[x]=\sum[L_g\bar{p}\bar{x}]=f(\sum g[px])$ with $\sum g[px]$ in $Q_{max}(R)G$. Hence f is onto. Resultly f is an isomorphism. Thus $Q_{max}(R)G$ is isomorphic to $Q_{max}(RG)$.

For a finite group G of automorphisms of a semiprime ring R, let $t=\sum g$ and $S=RG$. We note tR is a bi R^G-S module, the right action of S being $tr\sum r_g g=\sum t(rr_g)^g$. The left action of R^G is clear. Also R is a bi $S-R^G$ module, where $\sum r_g gr=\sum r_g r^{g^{-1}}$. In this case to consider the derived Morita context of the left S-module ${}_SR$, we note that $Hom({}_SR, {}_SR)=R^G$ and $Hom({}_SR, S)=tR$.

Lemma 3.6. The derived Morita context of ${}_SR$ is $\langle S, {}_SR_{R^G}, {}_{R^G}tR_S, R^G\rangle$, where pairings are $(\ ,\ )$: $tR\otimes_S R \to R^G$, $(ta, b)=tr(ab)$ and $[\ ,\ ]$: $R\otimes_{S^G} tR \longrightarrow S$: $[a, tb]=atb$.

Proof. Let p: $tR \longrightarrow Hom({}_SR, S)$ by $p(tr)=f$, where $f(r)=tr$. Then $Hom({}_SR, S)=tR$. And let $q: R^G \longrightarrow Hom({}_SR, {}_SR)$ by $q(r)=g$, where $g(r)=r$ for all r in R. Then $Hom({}_SR, {}_SR)=R^G$. Since $tc[a, tb]=tc(atb)=\sum gc(atb)=\sum(gca)tb=\sum(ca)^g tb=tr(ca)tb=(tc, a)tb$, and $[a, tb]c=(atb)c=a(tbc)=a(\sum gbc)=a(\sum(bc)^g)=atr(bc)=a(tb, c)$.

Therefore $\langle S, {}_SR_{R^G}, {}_{R^G}tR_S, R^G\rangle$ is the derived Morita context of ${}_SR$.

Now we prove the result of [8, Theorem 2] differently.

THEOREM 3.7. If the derived Morita context $\langle S, {}_SR_{R^G}, {}_{R^G}tR_S, R^G\rangle$ in nondegenerate, then $Q_{\max}(R)^G = Q_{\max}(R^G)$.

PROOF. Since $\langle S, {}_SR_{R^G}, {}_{R^G}tR_S, R^G\rangle$ is nondenerate, $\langle Q_{\max}(S), Q({}_SR_{R^G}), Q({}_{R^G}tR_S), Q_{\max}(R^G)\rangle$ is right normalized by Theorem 2.6. By Corollary 3.4 and Theorem 3.2, $Q_{\max}(S) = Q_{\max}(R)G$ and $Q({}_SR)$ is the left quotient module $Q_{\max}(R)$ over the ring $Q_{\max}(S)$. So we have $Q_{\max}(R^G) = Hom_{Q_{\max}(S)}(Q({}_SR), Q({}_SR))$ and $Hom_{Q_{\max}(R)G}(Q_{\max}(R), Q_{\max}(R)) = Q_{\max}(R)^G$.

As is elementary and well known, one can imbed a commutative integral domain in a field, being nothing else than the fractions created from the elements of the domain. O. Ore gave the appropriate conditions in order that this be possible for noncommutative rings without zero divisors. We shall give an account of this rather, more general situation below. But first a few definition are needed.

DEFINITION 3.8. An element in a ring R is said to be *regular* if it is neither a left nor right zero divisor in R.

DEFINITION 3.9. An extension ring $Q(R)$ of R is said to be *a left quotient ring for* R if:

1. every regular element in R is invertible in $Q(R)$.
2. every $x \in Q(R)$ is of the form $x = a^{-1}b$ where a, $b \in R$ and a is regular.

If $Q(R)$ is a left quotient ring of R we say that R is left order in $Q(R)$. In any ring R, for a nonempty subset

S of R let $l(S)=\{x\in R: xs=0$ for all $s\in S\}$. We call $l(S)$ the left annihilator of S and term a left ideal λ of R a left annihilator if $\lambda=l(S)$ for some appropriate S in R. We similarly define the right annihilator $r(S)$ of S and speak of a right ideal as a right annihilator.

DEFINITION 3.10. A ring R is said to be a *(left) Goldie ring* if:

1. R satisfies the ascending chain condition on left annihilators.
2. R contains no infinite direct sums of left ideals.

Clearly a left Noetherian ring, that is, one satisfying the ascending chain condition on left ideals is a Goldie ring. A ring R is said to be semiprime if it has no nonzero nilpotent ideals.

THEOREM 3.11 [Goldie]. Let R be a semiprime left Goldie ring. Then R has a left quotient ring $Q=Q(R)$ which is semisimple artinian.

There has been a great deal of interest in group of outer automorphism, i.e., automorphism g for which there does not exist a unit u such that $r^g=u^{-1}ru$ for all r in R. Let R be a semiprime ring with a finite group G of ring automorphisms of R. Let S denote the ring of quotients of R relative to the Gabriel filter which consists of all two sided ideals whose annihilator is 0. An automorphism g is called *X-outer* if $sr^g=rs$ for an s in S and for all r in R implies that $s=0$. The group is called *X-outer* if each g $(g\neq 1)$ in G is X-outer.

COROLLARY 3.12 [S. Montgomery]. If R is semiprime ring and a finite group G of ring automorphism of R is X-outer. Then R is right Goldie if and only if R^G is right Goldie.

PROOF. If R is right Goldie, then $Q_{\max}(R)$ is semi-simple Artinian and G acts on $Q_{\max}(R)$ as X-outer. So G is completely outer on $Q_{\max}(R)$ and hence $Q_{\max}(R)^G = Q_{\max}(R^G)$ is semi-simple Artinian. Thus R^G is semiprime right Goldie.

Conversely, suppose R^G is right Goldie. Then since the context $\langle S, R, Rt, R^G\rangle$ is nondegenerate, R^G is semiprime and hence $Q_{\max}(R^G) = Q_{\max}(R)^G$ is semisimple Artinian. Now by Amitsur [1], Muller [7] and Theorem 3.2, the maximal quotient context $\langle Q_{\max}(R)G, Q_{\max}(R), Q_{\max}(R)t, Q_{\max}(R)^G\rangle$ is also nondegenerate and hence $Q_{\max}(R)t_{Q_{\max}(R)^G} = Q_{\max}(R)_{Q_{\max}(R)}$ has finite Goldie dimension. Hence $Q_{\max}(R)$ is finitely generated over $Q_{\max}(R)^G$ and so $Q_{\max}(R)$ is Artinian. From the nondegeneracy of the maximal quotient context, the semi-primitivity of $Q_{\max}(R)G$ follows from $Q_{\max}(R)^G$ and so $Q_{\max}(R)$ is semiprimitive. Hence R is right Goldie.

LEMMA 3.13. If R is a right rationally complete, semiprime ring and $\langle S, R, Rt, R^G\rangle$ is nondegenerate. Then

(1) R is right self-injective iff R_{R^G} in injective.

(2) If R^G is right self-injective then $tr(R) = R^G$.

(3) If $tr(R) = R^G$ then R_{R^G} is finitely generated.

PROOF. (1) Suppose R_{R^G} is injective. Then $Hom_{R^G}(R, R) = S$ is injective because R_{R^G} is torsion free with respect to torsion theory induced from the trace ideal $tr(R)$ of Rt_{R^G}. Therefore R_R is injective. In a similar fashion since S is torsion free with respect to the hereditary torsion theory induced from the trace ideal RtR of R_S, the self-injectivity of S implies that $Hom_S(R_S, S_S)=Rt_{R^G}=R_{R^G}$ is injective.

(2) Suppose R^G is right self-injective. Then by the same reason as in (1), $Hom_{R^G}(Rt_{R^G}, R^G{}_{R^G})=R_S$ is injective. Hence R_S is an S-direct summand of S_S. Therefore R_S is projective and hence $tr(R)=R^G$.

(3) Let $\mathscr{A}_S=\{A\in Mod\text{-}S \mid A\longrightarrow Hom_S(RtR, A)$: bijective$\}$ and $\mathscr{A}_{R^G}=\{B\in Mod\text{-}R^G \mid B\longrightarrow Hom_{R^G}(tr(R), B)$: bijective$\}$. Then $\mathscr{A}_S$ and $\mathscr{A}_{R^G}$ are quotient categories of $Mod\text{-}S$ and $Mod\text{-}R^G$, respectively corresponding to hereditary torsion theories induced by trace ideals RtR and $tr(R)$. By Muller [7, Theorem 3], ~~two functors~~ $Hom_S(R_S, -)$ and $Hom_{R^G}(Rt_{R^G}, -)$ induces equivalences between $\mathscr{A}_S$ and $\mathscr{A}_{R^G}$. Let Λ denote quotient functors with respect to hereditary torsion theories induced by trace ideals. Then since $Hom_S(R_S, S)=Hom_S(R_S, S)=Rt_{R^G}$ the lattice of $\mathscr{A}_S$-subobject of S and $\mathscr{A}_{R^G}$-subobject of Rt_{R^G} are lattice isomorphic. Now to prove (3); suppose $tr(R)=R^G$. Then every R^G-submodule of Rt_{R^G} is $\mathscr{A}_{R^G}$-subobject. Now assume to the contrary that Rt_{R^G} is not finitely generated. Then there is a totally ordered set $\{I_\alpha\}$ of proper R^G-submodules of Rt_{R^G} with $\bigcup_\alpha I_\alpha=Rt$. Hence $\{Hom_{R^G}(R, I_\alpha)\}$ is a totally ordered set of right proper $\mathscr{A}_S$-subobject of S. Since $Hom_{R^G}(R, \bigcup_\alpha I_\alpha)=\bigcup_\alpha Hom_{R^G}(R, I)$, we have $\bigcup_\alpha Hom_{R^G}(R, I_\alpha)=Hom_{R^G}(R, Rt)=S$. But this is impossible because S_S is finitely generated. Therefore $Rt_{R^G}=R_{R^G}$ is finitely generated.

A ring R is called *G-Galois extension* of R^G if there are elements $a_1, a_2, \cdots, a_n$; $a_1^*, a_2^*, \cdots, a_n^*$ in R such that $\sum_{i=q}^{n} a_i a_i^{*g} = \delta_{1,g}$ for all g in G, where δ is the Kronecker delta.

S. U. Chase, D. K. Harrison and A. Rosenberg [2] have shown that R is a G-Galois extension of R^G if and only if R is a finitely generated projective R^G-module and the map j from RG to $End_{R^G}(R)$ defined by $j(xg)(y) = xy^g$ for x, y in R and g in G is a ring isomorphism.

THEOREM 3.14. If R is a von Neumann regular selfinjective ring and G is X-outer, then

(1) R is a G-Galois extension of R^G.
(2) R_{R^G} is injective.

PROOF: If I is an essential right ideal of R^G, then IR is an essential right ideal of R because G is X-outer and R is regular, self-injective. Hence R^G is nonsingular. Since $\langle S, R, Rt, R^G \rangle$ is nondegenerate, the nonsingularity of R^G implies those of $S = RG$, R_S and Rt_{R^G}. So S is regular. Since R_S is finitely generated, R_S is projective and hence $tr(R) = R^G$. Therefore by Lemma 3.8, R_{R^G} is finitely generated. On the other hand, since R^G is semi-prime, it is nonsingular and self-injective. Now since $\langle S, R, Rt, R^G \rangle$ is right normalized, R is a G-Galois extension of R^G.

(2) Since R_R is injective by Lemma 3.13, so is R_{R^G}.

References

[1] S. A. Amitsur, Rings of quotients and Morita contexts, J. Algebra, 17 (1971), 273-298.

[2] S. U. Chase, D. K. Harrison and A. Rosenburg, Galois theory and cohomology of commutative rings, Mem. Amer. Math. Soc.

52(1965).

[3] M. Cohen, A Morita context related to finte automorphism group of rings, Pacific J. Math. 98(1982), 37-54.

[4] J.W. Fisher and Susan Montgomery, Semiprime skew group rings, J. Algebra, 52. No. 1. (1978) 241-247.

[5] C.O. Kim and C. Huh, Subrings of normalizing extensions, Science J. Pusan National Univ. (1983), 19-23.

[6] K. Louden, Maximal quotient rings of ring extensions, Pacific J. Math. 62(1976), 487-486.

[7] B. Muller, The quotient category of a Morita context, J. Algebra, 28(1974), 389-407.

[8] T. Nakayama, Galois theory for general rings with minimun conditions, J. Math. Soc. Japan 1(1949), 203-216.

[9] J. Osterburg and J.K. Park, Morita contexts and quotient rings of fixed rings, Houston J. Math. 10(1984), 75-80.

[10] Bo Stenstrom, Rings and module of quotients, Lecture Notes in Math. No. 237, Springer, 1971.

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