PUSAN KYŎNGNAM MATHEMATICAL JOURNAL
Vol. 3, 55—60, 1987

COHERENT POLYNOMIAL RINGS OVER REGULAR P.I. RINGS

CHANG WOO HAN

1. Introduction

J.P. Soublin [11] showed that the analog of the Hilbert basis theorem fails for coherent rings, but the ring of polynomials in an indeterminate over a commutative von Neumann regular ring with identity is coherent. G. Sabbagh [10] has proved that the ring of polynomials in any finite number of indeterminates over a commutative von Neumann regular ring with identity is coherent.

In this paper, the following result will be studied if R is a von Neumann regular P.I. ring then $R[x_1, x_2, \cdots, x_n]$ is a coherent ring. Also, we obtain the following corollaries; (1) If R is a commutative von Neumann regular ring, then $R[x_1, x_2, \cdots, x_n]$ is a coherent ring, see G. Sabbagh [10]. (2) Let R be a P.I. ring. Then R is von Neumann regular if and only if $R[x]$ is semihereditary.

2. Self-injective ring and Azumaya algebra

We introduce P.I. rings and Azumaya algebras that will be used in the study of our main results. It is known that, since singular ideals of semiprime P.I. ring is zero [3], maximal quotient rings of semiprime P.I. rings are regular with identity [2], self-injective [2] and P.I. [7]. By combining the above facts and Armendariz's two decomposition theorems [1], we obtain the fact that the maximal quotient ring of a regular P.I. ring is Azumaya.

THEOREM 2.1[1]. If R is a regular self-injective ring with a P.I., then $R \cong \prod_{\lambda \in \Lambda} R_\lambda$ where each R_λ is a matrix ring over a strongly self-injective ring with a P.I..

THEOREM 2.2[1]. Let R be a regular self-injective ring with a P.I., then $R = \oplus \sum_{i=1}^{k} R_i$ where R_i is either zero or else each R_i is a product of regular self-injective rings each of which is stable of degree i, $1 \leqq i \leqq k$.

The conclusion of Theorem 2.2 obtains following.

COROLLARY 2.3. If R is a regular P.I. ring, then the quotient ring $Q(R)$ of R is an Azumaya algebra.

PROOF. If R is a regular P.I. ring, then $Q(R)$ is a regular self-injective P.I. ring. By Theorem 2.2,

$$Q(R) = \oplus \sum_{i=1}^{k} Q_i(R)$$

where $Q_i(R)$ is a product of regular self-injective rings each of which is stable of degree $1 \leqq i \leqq k$. Hence each $Q_i(R)$ is stable unital semiprime P.I. ring. By Procesi, stable unital semiprime P.I. rings are Azumaya algebras. Therefore each $Q_i(R)$ is an Azumaya algebra. Thus $Q(R) = \oplus \sum_{i=1}^{k} Q_i(R)$ is an Azumaya algebra.

3. Coherent Ring

By Sabbagh, the ring of polynomials in any number of indeterminates over a commutative von Neumann regular ring with identity is coherent. In this section, we obtain similar results for a von Neumann regular P.I. ring.

THEOREM 3.1 [6]. R is a subdirect of the rings S_i, $i \in I$, if and only if $S_i \cong R/K_i$, K_i an ideal of R, and $\bigcap_{i \in I} K_i = 0$.

THEOREM 3.2 [6]. Every left R-module is flat if and only if R is regular.

THEOREM 3.3. Let R be a commutative von Neumann regular ring and $S = \Pi R/P_i$, P_i a prime ideal of R. Then $S[x]$ is faithfully flat over $R[x]$.

PROOF. First note that as left S-module

$$S[x] \cong S \otimes_R R[x].$$

Next note that, for any left $S[x]$-module M, there are the following left S-module isomorphisms

$$S[x] \otimes_{R[x]} M \cong S \otimes_R R[x] \otimes_{R[x]} M \cong S \otimes_R M.$$

Now consider an exact sequence of $R[x]$-module

$$0 \to M \to N.$$

Then, since S_R is flat, the following diagram is commutative and its columns are isomorphisms

$$\begin{array}{ccccc} 0 & \longrightarrow & S \otimes_R M & \longrightarrow & S \otimes_R N \\ & & \wr\| & & \wr\| \\ 0 & \longrightarrow & S[x] \otimes_{R[x]} M & \longrightarrow & S[x] \otimes_{R[x]} N \end{array} .$$

Thus $S[x]$ is flat. Now to show that $S[x]$ is faithfully flat as $R[x]$-module, let $S[x] \otimes_{R[x]} M = 0$. Then $S \otimes_R M = 0$ so that $M = 0$. Hence $S[x]$ is faithfully flat over $R[x]$.

COROLLARY 3.4. Let R be a commutative von Neumann regular ring. Then $S[x_1,x_2,\cdots,x_n]=\Pi R/P_i[x_1,x_2,\cdots,x_n]$ is faithfully flat over $R[x_1,x_2,\cdots,x_n]$.

THEOREM 3.5. Let R be a regular P.I. ring. Then the maximal quotient ring $Q(R)$ of R is faithfully flat over R.

PROOF. Since $Q(R)$ is regular, it is flat. Let M be a left R-module. Suppose that $Q(R)\otimes_R M=0$, using the flatness of M as a left R-module $M\cong R\otimes_R M\leqq Q(R)\otimes_R M=0$. Thus $M=0$.

COROLLARY 3.6. If R is regular P.I. ring, then $Q(R)[x_1,x_2,\cdots,x_n]$ is faithfully flat over $R[x_1,x_2,\cdots,x_n]$.

THEOREM 3.7 [10]. Let R be a subring of the right coherent ring S such that S is faithfully left flat over R. Then R is a right coherent ring.

THEOREM 3.8. If R is a regular P.I. ring, then $R[x_1,x_2,\cdots,x_n]$ is a coherent ring.

PROOF. Since the maximal quotient ring $Q(R)$ of R is a regular self-injective P.I. ring, $Q(R)=\sum_{i=1}^{n} Q_i(R)$, where each $Q_i(R)$ is an Azumaya algebra. Therefore,

$$Q(R)[x_1,x_2,\cdots,x_n]=\sum_{i=1}^{n} Q_i(R)[x_1,x_2,\cdots,x_n]$$

since $Q_i(R)\cong Mat_n(A)$, where A is regular in which every idempotent is a central idempotent. For each i,

$$Q_i(R)[x_1,x_2,\cdots,x_n]=Mat_n(A[x_1,x_2,\cdots,x_n]).$$

Now we show that $A[x_1,x_2,\cdots,x_n]$ is coherent, since matrix ring with coefficient in a coherent ring is also coherent [9]. Let $\{P_i|i\in I\}$ be a set of all prime ideals of A. Then $\bigcap_i P_i=0$. Therefore R is a subdirect product of division rings $\{A/P_i|i\in I\}$. So $\Pi A/P_i$ is faithfully flat over A. It follows that $\Pi A/P_i[x_1,x_2,\cdots,x_n]$ is faithfully flat over $A[x_1, x_2,\cdots,x_n]$. Since A/P_i is a division ring, $A/P_i[x_1,x_2,\cdots,x_n]$ is coherent. Hence $\Pi A/P_i[x_1,x_2,\cdots x_n]$ is coherent. Thus

$$\begin{array}{ccc}\Pi A/P_i[x_1,x_2,\cdots,x_n]\cong(\Pi A/P_i)[x_1,x_2,\cdots,x_n]: \text{right coherent} \\ \cup\!\mid \quad \text{faithfully flat} \quad \cup\!\mid \\ A[x_1,x_2,\cdots,x_n] \longrightarrow A[x_1,x_2,\cdots,x_n]: \text{coherent.}\end{array}$$

Therefore $R[x_1,x_2,\cdots,x_n]$ is coherent.

The corollary of ours is proved by Sabbagh [21].

COROLLARY 3.9. If R is commutative regular, then $R[x_1, x_2,\cdots,x_n]$ is coherent.

The weak dimension of an R-module will be denoted by $WD(R)$. Regular rings are characterized by nullity of the weak dimension. From properties of semihereditary and coherent ring, we see that a ring is left semihereditary if and only if it is left coherent and has weak dimension at most one.

COROLLARY 3.10. Let R be a P.I. ring. Then R is von Neumann regular if and only if $R[x]$ is semihereditary.

PROOF. In the proof of Theorem 3.8, we see that for the maximal quotient ring $Q(R)$ of a ring R, $Q(R)[x]$ is coherent. By Theorem 3.7 and Lemma 3.8, $R[x]$ is coherent. For every ring R, $WD(R)\leqq WD(R[x])\leqq 1+WD(R)$ since $WD(R)=0$, $0\leqq WD(R[x])\leqq 1$. If $WD(R[x])=0$, then $R[x]$ is semi-simple Artinian. Hence $WD(R[x])\leqq 1$.

References

[1] E.P. Armendariz and S.A. Steinberg, Regular self-injective rings with a polynomial identity, Trans. Amer. Math. Soc. 190 (1974), 417-424.

[2] C. Fairh, Lectures on injective modules and quotient rings, Lecture Notes in Math. No. 49, Springer-Verlag, Berlin and New York, 1967.

[3] J.W. Fisher, Structure of semiprime P.I. rings, Proc. Amer. Math. Soc. 39(1973), 465-467.

[4] M.E. Harris, Some results on coherent rings, Proc. Amer. Math. Soc. 17(1966), 474-479.

[5] C.U. Jensen, On homological dimensions of rings with countably generated ideals, Math. Scand. 18(1966), 97-105.

[6] J. Lambek, Lectures on Rings and Modules, Chelsea Publ. Co. New York (1976).

[7] W.S. Martindale III, On semiprime P.I. rings, Pr>c. Amer. Math. Soc. 40(1973), 365-369.

[8] D.G. McRae, Homological dimensions of finitely presented modules, Math, Scand. 28(1971), 70-76.

[9] J. Rotman, An Introduction to Homological Algebra, Academic Press, New York, 1979.

[10] G. Sabbagh, Coherent of polynomial rings and bounds in polynomial ideals, J. Algebra 31(1974), 449-507.

[11] J.P. Soublin, Anneux coherents, J. Algebra 15(1970), 455-472.

Department of Mathematics
Dong-A University
Pusan, 600-02
Korea

Received March 2, 1987