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PRIME DUAL IDEALS IN TANAKA ALGEBRAS

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1. Introduction

K. Iseki [6] has introduced the notion of a BCK-algebra which is an algebraic formulation of a propositional calculus. We refer to Iseki [7], [8], and [9] for certain basic properties of these algebras. The ideals and their properties were studied by K. Iseki and S. Tanaka [10]. Elias Deeba [5] has introduced the *notion of dual ideals in BCK-algebras.* In [1], B. Ahmad has given a characterization of prime dual ideals in Tanaka algebras. In this note, we obtain some properties of prime dual ideals in Tanaka algebras. We recall that a set X is said to be a Tanka algebra [9] if the following conditions are satisfied:

(1) $(X, \leq)$ is a partially ordered set with least element 0,

(2) $(x*y)*z=(x*z)*y$,

(3) $x*(x*y)=y*(y*x)$,

where $x\leq y$ means $x*y=0$.

S. Tanaka proved that the algebra X is a semilattice with respect to $x\wedge y$ which is defined by $y*(y*x)$, and X is a BCK-algebra, i.e. $(x*y)*(x*z)\leq z*y$ holds in X (See S. Tanaka [11], [12]).

In a BCK-algebra, the notion of a dual ideal has been defined in [5] as follows:

DEFINITION. A non-empty subset D of X is a *dual ideal* in X if the following conditions are satisfied:

(1) $x \in D$, $x \leq y$ imply $y \in D$.

(2) $x \in D$, $y \in D$ imply there exists an element $z \in D$ such that $z \leq x$, $z \leq y$.

In [1], B. Ahmad has defined a prime dual ideal as follows:

DEFINITION. A dual ideal P in a Tanaka algebra X is called a *prime dual ideal* if for any x, y, $x \vee y \in P$ implies $x \in P$ or $y \in P$.

2. Main Results

LEMMA 1. Let $\{P_i\}_{i \in I}$ be a non-empty family of prime dual ideals in a bounded Tanaka algebra X. If the family is totally ordered by set inclusion, then both $\bigcup_i P_i$ and $\bigcap_i P_i$ are prime dual ideals.

PROOF. To prove $\bigcup_i P_i$ (put P') is a prime dual ideal, suppose $x \in P'$ and $x \leq y$ for every $x, y \in X$. Then we have $x \in P_i$ and $x \leq y$ for some $i \in I$, which imply $y \in P_i$ for some i. This means that $y \in P'$. Assume that x and y are in P'. Then $x \in P_i$ and $y \in P_j$ for some i, j. If $P_i \subset P_j$ then $x, y \in P_j$, and hence there exists an element $z \in P_j \subset P'$ such that $z \leq x$ and $z \leq y$. If $P_j \subset P_i$ then $x, y \in P_i$, and therefore there exists $z \in P_i \subset P'$ such that $z \leq x$ and $z \leq y$. Thus in any case there exists an element $z \in P'$ with $z \leq x$ and $z \leq y$. It follows that P' is a dual ideal. Next suppose that $x \vee y \in P'$ and $x \notin P'$. Then $x \vee y \in P_i$ and $x \notin P_i$ for some i, which imply $y \in P_i \subset P'$. Therefore P' is a prime dual ideal.

To prove $\bigcap_i P_i$(put P'') is a prime dual ideal, we first assume that $x \in P''$ and $x \leq y$ for all $x, y \in X$. Then $x \in P_i$ and $x \leq y$ for all $i \in I$. This imply that $y \in P_i$ for all i, and hence $y \in P''$. If x and y are in P'' then $x, y \in P_i$ for all i. Then there exists $z \in P_i$ such that $z \leq x$ and $z \leq y$ for all i. It follows that $z \in P''$ with $z \leq x$ and $z \leq y$. Thus P'' is a dual ideal of X. Now suppose that $x \vee y$ belongs to P'' but $x \notin P''$. Then it is possible to choose i with $x \vee y \in P_i$ but $x \notin P_i$. Then $y \in P_i$. Finally let j be an arbitrary element of I. If $P_i \subset P_j$ then $y \in P_j$. On the other hand, if $P_j \subset P_i$ then $x \vee y \in P_j$ while $x \notin P_j$. Consequently $y \in P_j$. Thus in any case $y \in P_j$ and hence $y \in P''$. Therefore P'' is a prime dual ideal and the proof is complete.

PROPOSITION 2. Let D be a dual ideal of a bounded Tanaka algebra X and let P be a prime dual ideal containing D. Then P contains a prime dual ideal which contains D and has no smaller prime dual ideal containing D.

PROOF. Denote by $\mathscr{P}$ the set of all prime dual ideals which contain D and are contained in P. Then $\mathscr{P}$ is not empty. Define a relation $\leq$ on $\mathscr{P}$ by $P' \leq P''$ if and only if $P'' \subset P'$ for all P', $P'' \in \mathscr{P}$. Then $(\mathscr{P}, \leq)$ is a partially ordered set. Let S be a non-empty totally ordered subset of $\mathscr{P}$. By the above Lemma, the intersection of all members of S is a prime dual ideal $\bar{P}$, say. This certainly contains D and is contained in P. Consequently $\bar{P} \in \mathscr{P}$. Since $\bar{P} \subset P'$ for all $P' \in S$, we have $P' \leq \bar{P}$ for every $P' \in S$. Thus $\bar{P}$ is an upper bound for S. By Zorn's Lemma, $\mathscr{P}$ contains a maximal element P^*, and hence P^* is a prime dual ideal and $D \subset P^* \subset P$. Suppose now that P^{**} is a prime dual ideal satisfying $D \subset P^{**} \subset P^*$. Then $P^{**} \in \mathscr{P}$ and $P^* \leq P^{**}$. By the maximality of P^*, we have $P^*=P^{**}$, which completes the proof.

LEMMA 3. Let X be a bounded and implicative BCK-algebra and D be a dual ideal of X. Then D is maximal dual implies D is a prime dual ideal.

PROOF. See [3], p. 650.

PROPOSITION 4. Let X be a bounded and implicative Tanaka algebra, A an ideal of X, and let D be a dual ideal of X such that $D\cap A=\phi$. Then X contains a prime dual ideal which contains D and disjoint from A.

PROOF. Let $\mathscr{D}$ be the set of all dual ideals which contain D and disjoint from A. $\mathscr{D}$ is non-empty because $D\in\mathscr{D}$. We shall show that $\mathscr{D}$ is inductively ordered by inclusion. To this purpose, let $\mathscr{D}'$ be a totally ordered non-empty subset of $\mathscr{D}$. Let E be the union of all dual ideals in $\mathscr{D}'$. Then, by Lemma 1, E is a dual ideal. Also E contains D and disjoint from A, which implies that $E\in\mathscr{D}$. Moreover, it is clear that E is an upper bound of $\mathscr{D}'$. By Zorn's Lemma, $\mathscr{D}$ has a maximal element, say P. It follows from Lemma 3 that P is a prime dual ideal.

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