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ON THE JORDAN STRUCTURE IN OPERATOR ALGEBRAS

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1. Introduction

The study of JB-algebras was initiated by Alfsen, Shultz and Stømer [3], even though earlier approaches have been made by von Neumann and Segal. In [3], the study of JB-algebras can be reduced to the study of Jordan algebras of self-adjoint operators on a Hilbert space and M_3^8.

The purpose of this note is to show Jordan-Banach algebra versions of some facts about C*-algebras by some modifications. In section 2, we give the formal definitions of JB-algebras and JB*-algebras and some known results. In section 3, we study projections and ideals in JB-algebra. In section 4, we study the multipliers of JB-algebras.

2. Preliminaries

A *Jordan Banach algebra* is a real Jordan algebra A equipped with a complete norm satisfying

$$\|a\circ b\|\leq\|a\|\ \|b\|,\quad a,\ b\in A.$$

A *JB-algebra* is a Jordan Banach algebra A in which the norm satisfies the following two additional conditions for $a,\ b\in A$:

(i) $||a^2||=||a||^2$

(ii) $||a^2||\leq||a^2+b^2||$.

Examples of JB-algebras are JC-algebras, i.e., the norm closed. Jordan algebras of self-adjoint operators on a complex Hilbert space, and the exceptional M_3^8 consisting ofall Hermitian 3×3 matrices over the Cayley number.

Note that an associative JB-algebra can be realized as the self-adjoint part of a commutative C*-algebra [15]. In finite dimension, JB-algebras are precisely the formally real Jordan algebras. However this is not true in infinite dimensional JB-algebras [13].

A JB-algebra which is also a Banach dual space is said to be a *JBW-algebra*. Then the second dual, A^{**} of a JB-algebra A is a unital JB-algebra and moreover, it is a JBW-algebra in the Arens product which contains A [14]. A special case of this is already known; if A is a JC-algebra then A^{**} is isomorphic to a JC-algebra [12].

The reader is referred to [3, 4, 12, 13] for properties of JB-algebras. The complex analogue of JB-algebras are the JB*-algebras (Kaplansky's Jordan C*-algebras), introduced by Kaplansky, who first presented it at a lecture for the Edinburgh Mathematical Society in July 1976.

A *JB-algebra* is a complex Jordan Banach algebra $\mathscr{A}$ with an involution * such that for all $x\in\mathscr{A}$,

$$||\{x,\ x^*,\ x\}||=||x||^3 \text{ holds.}$$

For example, every C*-algebra A is a JB-algebra in the Jordan product. The second dual, $\mathscr{A}^{**}$ of a JB*-algebra $\mathscr{A}$ with the Arens product, is a unital JB*-algebra [20].

It is known that the set of self-adjont elements of a unital JB*-algebra forms a unital JB-algebra, while, conversely, the complexification of a unital JB-algebra in a suitable norm is a JB*-algebra [17]. In [20], this also holds for non-unital JB-algebras. Therefore JB-algebras and JB*-algebras are in a one-to-one correspondence.

A *Jordan W*-algebra* is a unital JB*-algebra which is the dual of a complex Banach space. In [11], it is shown that the self-adjoint part of a Jordan W*-algebra is a JBW-algebra and the complexification of a JBW-algebra is a Jordan W*-algebra. The general theory of JB*-algebras can be found in [11, 17, 19, 20].

3. Projections and Ideals in JB-algebras

If $p^2=p$ then p is called an *idempotent*. An idempotent in a JB-algebra will be called a *projection*.

Let A be a JB-algebra and let a, b, c be elements of A. The *Jordan triple product* $\{a, b, c\}$ is defined by

$$\{a, b, c\} = (a \circ b) \circ c + a \circ (b \circ c) - (a \circ c) \circ b$$

and for $a \in A$, U_a and L_a are defined by

$$U_a b = \{a, b, a\}, \qquad L_a b = a \circ b \text{ for } b \in A.$$

Note that if A is a JC-algebra then $\{a, b, c\} = \frac{1}{2}(abc + cba)$.

Recall that two projections p and q are said to be *orthogonal* if $p \circ q = 0$.

LEMMA 3.1. Let p and q be projections in the JB-algebra A. Then the followings are equivalent.

(i) $pq=0$ (ii) $p \circ q = 0$ (iii) $\{p, q, p\} = 0$ (iv) $p+q$ is a projection.

PROOF. By [13, Lemma 4.2.2] and easy calculation.

Let A and B be JC-algebras. We call a linear map ϕ from A into B is a *Jordan homomorphism* if $\phi(a\circ b)=\phi(a)\circ\phi(b)$ for all a, $b\in A$ and ϕ takes the identity into the identity.

PROPOSITION 3.2. Let p and q be orthogonal projections of JC-algebra A and ϕ is a Jordan homomorphism. Then $\phi(\{p,x,q\})=\{\phi(p),\phi(x),\phi(q)\}$ holds for all $x\in A$.

PROOF. Since $2(p\circ x)\circ q=\{p,x,q\}$ and $\phi(p)\phi(q)=0$ by Lemma 3.1 we have

$$\begin{aligned}\phi(\{p,x,q\})&=\phi(2(p\circ x)\circ q)=2(\phi(p)\circ\phi(x))\circ\phi(q)\\&=\{\phi(p),\phi(x),\phi(q)\}.\end{aligned}$$

The following is a slight modification of [7, Proposition 1.5.8].

PROPOSITION 3.3. Let A and B be JC-algebras and ϕ is a Jordan homomorphism from A into B. If p is a projection of A, then $\phi(p)$ is a projection of B.

PROOF. We get $\{\phi(p)\}^2=\phi(p)\circ\phi(p)=\phi(p\circ p)=\phi(p^2)=\phi(p)$ since p is a projection. Hence $\phi(p)$ is a projection of B.

Recall that elements a, b in a JB-algebra A are said to *operator commute* if $L_aL_b=L_bL_a$. i.e., if $(a\circ c)\circ b=a\circ(c\circ b)$ for all c in A. If p is a projection in A than a and p operator commute if and only if $L_pa=U_pa$ or $a=U_pa+U_{e-p}a$.

A projection p in A is said to be *central* if p operator commutes with every element of A.

REMARK. Central projections can be used to construct more general ideals. For example, if A is a JB-algebra, B a JB-subalgebra of A and p a central projection in A, then the set of all b in B such that $p\circ b=0$ is an ideal in B (in fact it is a Jordan ideal). For, let $J=\{b\in B | p\circ b=0\}$. If $a\in J$, $c\in B$, then $p\circ(a\circ c)=(p\circ a)\circ c=c\circ(p\circ a)=0$. Hence $a\circ c\in J$.

A subspace J of a JB-algebra A is said to be a *Jordan ideal* in A if $L_a b\in J$ whenever $a\in J$, $b\in A$. A linear subspae J of A is a Jordan ideal if and only if $aba\in J$ whenever $a\in A$ and $b\in J$ [12]. Note that Jordan ideals correspond to two-sided ideals in the following sense; A norm closed self-adjoint complex subspace $\mathcal{J}$ of a C*-algebra $\mathcal{A}$ is a two-sided ideal if and only if its self-adjoint part $\mathcal{J}_{sa}$ is a Jordan ideal of $\mathcal{A}_{sa}$. This can be seen easily by considering the weak*-closure in $\mathcal{A}^{**}$ of $\mathcal{J}$ and using [8, Theorem 2.3], or by [12, Theorem 2].

A subspace J is said to be a *quadratic ideal* in A if $U_a b\in J$ whenever $a\in J$, $b\in A$. Note that every Jordan ideal is a quadratic ideal.

LEMMA 3.4. Let J be a Jordan ideal in a JB-algebra A. Then A/J with its natural Jordan product and quotient norm is a JB-algebra.

Let $\mathcal{A}$ be a JB*-algebra with self-adjoint part A. A Jordan ideal $\mathcal{J}$ of $\mathcal{A}$ is said to be a **-ideal* if, whenever $z\in\mathcal{J}$ then $z^*\in\mathcal{J}$. Let J be the self-adjoint part of a norm closed ideal $\mathcal{J}$ of $\mathcal{A}$, then $\mathcal{J}=J+iJ$ and J is a norm closed ideal of A.

THEOREM 3.5 [17]. Let $\mathcal{A}$ be a JB*-algebra. Let $\mathcal{J}$ be a closed *-ideal. Then $\mathscr{A}/\mathscr{J}$, when equipped with the quotient norm, is a JB*-algebra. Furthermore, if J is the self-adjoint part of $\mathscr{J}$, then the self-adjoint part of $\mathscr{A}/\mathscr{J}$ is isometrically isomorphic to A/J.

REMARK. The self-adjoint part of Jordan *-ideals is precisely the Jordan ideal in the unital JB-algebra A which is the self-adjoint part of $\mathscr{A}$.

LEMMA 3.6. If A is a JB-algebra, then every weak *-ideal J of A^{**} is of the form $U_p(A^{**})$ for a central projection $p \in A^{**}$.

PROOF. By [3, Lemma 9.1] J will contain an increasing approximate identity $\{U_\alpha\}$, i.e., $0 \leq U_\alpha \leq 1$, $\alpha \leq \beta$ implies $U_\alpha \leq U_\beta$ and $\|U_\alpha \circ a - a\| \to 0$ for all $a \in J$. Since $A^{**} = \tilde{A}$, A^{**} is monotone complete; Let p be the least upper bound of $\{U_\alpha\}$ in A^{**}. Then by [3, Theorem 3.10], $U_\alpha \to p$ strongly. It follows that $p \in J$ and $p^2 = p$ is an identity for J and this is also the greatest projection in J. Since J is an ideal,

$$U_p(A^{**}) \subseteq J = U_p(J) \subseteq U_p(A^{**}),$$

which shows $J = U_p(A^{**})$. Furthermore, if $s^2 = 1$ and $s \in A^{**}$, then $U_s p$ is a projection in J and so $U_s p \leq p$. Since $U_s^2 = I$, by positivity of the map U_s, we have

$$p = U_s^2 p \leq U_s p \leq p \quad \text{so} \quad U_s p = p.$$

Since this holds for every symmetry, by [3, Lemma 5.3] p is central.

THEOREM 3.7. If p is a central projection in a JB-algebra A, then $U_p A$ is a Jordan ideal in A. Conversely, if p is a projection in A such that U_pA is a Jordan ideal then p is central.

PROOF. If p is a central projection in A and then, for $a\in A$, $L_pa=U_pa$. Therefore, for $b\in U_pA$,

$$b\circ a=L_ba=L_bL_pa=L_pL_ba=U_p(b\circ a) \text{ and } b\circ a\in U_pA.$$

It follows that U_pA is a Jordan ideal. Conversely, if $a\in A$ we must have $p\circ a\in U_pA$, thus $U_p(p\circ a)=p\circ a$. This implies $U_pa=L_pa$. Hence p is central.

4. Multipliers of JB-algebras

The concept of the multiplier algebra of a C*-algebra has been extended to JB-algebra by Edwards [10]. An element b in a second dual A^{**} of a JB-algebra A is said to a *multiplier* if, for each $a\in A$, $L_ab\in A$.

The set $M(A)$ of multipliers of the JB-algebra A is a unital JB-algebra and is the largest JB-subalgebra of A^{**} of A in which A is a Jordan ideal [10].

LEMMA 4.1. The JB-algebra A possesses an approximate identity.

PROPOSITION 4.2. If B is a JB-subalgebra of JB-algebra A containing an approximate identity for A, and operator commute, then $M(B)\subset M(A)$.

PROOF. Let $\{u_j\}$ be approximate identities for A contained in B. For $a\in A$ and $b\in M(B)$, $a\circ b=(\lim a\circ u_j)\circ b=a\circ(\lim u_j\circ b)\in A$ since $u_j\circ b\in B$ and operator commute. Hence $b\in M(A)$. Thus $M(B)\subset M(A)$.

For a JB-algebra A, A^+, the set of squares of elements of A, is a positive cone which generates A. A JB-subalgebra B of A is said to be an *hereditary JB-subalgebra* if whenever $0 \leq a \leq b$ with $a \in A$ and $b \in B$ then $a \in B$.

LEMMA 4.3 [5]. Let A be a JB-algebra and J be an hereditary JB-subalgebra of A. Then

(i) The abelian elements of A form an hereditary and norm closed set.

(ii) Each abelian element of J is an abelian element of A.

LEMMA 4.4. Every non-zero closed quadratic ideal in the multiplier algebra $M(A)$ of the JB-algebra A has non-zero intersection with A.

PROOF. Let J be a non-zero closed quadratic ideal in $M(A)$ and let b be a non-zero element of the positive cone J^+ in J. It follows from [8] that $b^{1/2}$ is also an element of J^+.

For each element $a \in A$,

$$U_{b^{1/2}}a = 2(L_{b^{1/2}})^2 a - L_b a$$

is an element of A since both b and $b^{1/2}$ are elements of $M(A)$. Let $\{u_j\}$ be approximate identities for A. Then $\{U_{b^{1/2}} u_j\}$ is a bounded increasing net in A which possesses a least upper bound in A^{**}. It follows from [14, Lemma 2.2] that this least upper bound is b. Therefore, for some j,

$$U_{b^{1/2}}u_j \neq 0 \text{ and } 0 \leq U_{b^{1/2}}u_j \leq b.$$

Hence the positive cone J^+ in J is a closed face of the cone $M(A)^+$ and it follows that $U_{b^{1/2}}u_j$ is an element of $J \cap A$.

The following theorem is a Jordan Banach algebra version of C*-algebra case [2, Proposition 2.3].

THEOREM 4.5. Each non-zero hereditary JB-subalgebra of $M(A)$ has a non-zero intersection with A.

PROOF. By Lemma 4.4 and by the fact that norm-closed quadratic ideals of JB-algebra A are precisely the hereditary JB-subalgebras of A.

A Jordan ideal J in a JB-algebra A is said to be *essential* in A if every non-zero closed Jordan ideals in B has non-zero intersection with J.

THEOREM 4.6 [10]. (i) The JB-algebra A is essential Jordan ideal in its multiplier algebra $M(A)$. (ii) If the JB-algebra A is ~~an essential Jordan~~ ideal in a ~~JB-algebra~~ B then there exists a Jordan isomorphism from B into $M(A)$ which is the identity mapping on A.

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