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ON A CHARACTERIZATION BY CONDITIONAL EXPECTATIONS

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1. Introduction

The exponential distribution as a failure model has wide applicability. A well known characterization of an exponential random variable T with (scale) parameter α, i.e.,

$$P(T\leq t)=\begin{cases}1-\exp\{-t/\alpha\} & \text{for } t\geq 0\\ 0 & \text{for } t<0\end{cases} \tag{1.1}$$

is given by its lack of memory property (Feller [1, p.8]), viz.,

$$P(T>t+s\mid T>s)=P(T>t) \text{ for all } s,t\geqq 0. \tag{1.2}$$

In [3, Theorem 1, p.1257], Shanbhag characterized the exponential distribution (1.1) by means of conditional expectation, namely,

$$E(T\mid T>y)=y+E(T) \text{ for all } y>0. \tag{1.3}$$

He then pointed out that the characterizing property (1.3) is more useful than the lack of memory property (1.2) in practical problems because "the assumption of the knowledge of expected values is more reasonable than the assumption of the knowledge of probability distributions" [3, p.1257].

In [2], Hamdan generalized Shanbhag's result to include a general class of absolutely continuous distributions such as the uniform, exponential and Weibull distributions.

In this note, by establishing a simple characterization of the exponential distribution, we show that the seemingly more general theorem of Hamdan [2, p.498] turns out to be equivalent to that of Shanbhag [3, Theorem 1, p.1257] in the sense that either one can be obtained from the other and that each can be established independently of the other.

2. The main result

We shall now prove the following theorem.

THEOREM. *A random variable T has cumulative distribution function*

$$P(T\leq t)=\begin{cases}1-\exp\{-h(t)/h(b)\} & \text{for } t\in[\alpha,\beta)\\ 0 & \text{for } t\notin[\alpha,\beta)\end{cases} \tag{2.1}$$

where b is a positive constant and h is a strictly increasing function from $[\alpha,\beta)$

onto $[0, \infty)$, *if and only if the random variable* $h(T)$ *is exponentially distributed with (scale) parameter* $h(b)$.

PROOF. The fact that h is strictly increasing shows that it is a $1-1$ function. Since $h : [\alpha, \beta) \to [0, \infty)$ is $1-1$ and onto, its inverse $h^{-1} : [0, \infty) \to [\alpha, \beta)$ exists and is also strictly increasing and onto.

To prove the necessity of the condition, suppose that (2.1) holds. If $s \in [0, \infty)$, then, clearly, $h(T) \le s$ if and only if $T \le h^{-1}(s)$. Moreover, $s \in [0, \infty)$ if and only if $h^{-1}(s) \in [\alpha, \beta)$. Thus,

$$\begin{aligned} P(h(T) \le s) &= P(T \le h^{-1}(s)) \\ &= \begin{cases} 1-\exp\{-h \circ h^{-1}(s)/h(b)\} & \text{for } s \in [0, \infty) \\ 0 & \text{for } s \notin [0, \infty) \end{cases} \\ &= \begin{cases} 1-\exp\{-s/h(b)\} & \text{for } s \ge 0 \\ 0 & \text{for } s < 0, \end{cases} \end{aligned}$$

i.e., $h(T)$ is exponentially distributed with parameter $h(b)$.

To prove the sufficiency of the condition, suppose that $h(T)$ is exponentially distributed with parameter $h(b)$. Then, for each $t \in [\alpha, \beta)$, it is clear that $T \le t$ if and only if $h(T) \le h(t)$ and that $t \in [\alpha, \beta)$ if and only if $h(t) \in [0, \infty)$. Thus,

$$P(T \le t) = P(h(T) \le h(t))$$

whence (2.1) follows, in view of (1.1) with $\alpha = h(b)$.

3. Conclusion

It is now an easy matter to derive Hamdan's theorem [2] from Shanbhag's [3, Theorem 1, p. 1257] via the above characterization, by virtue of the fact that

$$T > y \text{ if and only if } h(T) > h(y)$$

for any $y \in [\alpha, \beta)$.

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