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ON NA-CONTINUOUS FUNCTIONS

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1. Introduction

The purpose of this paper is to introduce a new class of functions named na-continuous functions. This class is stronger than the class of super-continuous functions due to Munshi [10], and weaker than the class of strongly continuous functions due to S. P. Arya [1]. In section 2, we obtain several characterizations of na-continuous functions. Section 3 is related to basic properties of na-continuous functions. In the last section, we investigate the relationships among na-continuity and several strong forms of continuity.

Throughout this paper, spaces always mean topological spaces and $g: X \to Y$ denotes a function of a space X into a space Y. Let S be a subset of a space X. The closure and the interior of S are denoted by $\mathrm{cl}(S)$ and $\mathrm{int}(S)$, respectively. A subset S is said to be regular open (resp. regular closed) if $\mathrm{int}(\mathrm{cl}(S)) = S$ (resp. $\mathrm{cl}(\mathrm{int}(S)) = S$). A point x in a space X is said to be a δ-cluster point of a subset S of X [15] if $S \cap U \neq \phi$ for each regular open set U of X containing x. A subset S is called δ-closed if all δ-cluster points of S are contained in S. Complement of a δ-closed set is called δ-open. Intersection of all δ-closed sets containing S is called δ-closure of S and denoted by $\delta\mathrm{cl}(S)$.

O. Njastad defined an α-set in a space as a set S such that $S \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(S)))$. S. N. Maheshwari defined a feebly open set as a set S such that therer exists an open set O such that $O \subset S \subset s\mathrm{cl}(O)$, where $s\mathrm{cl}(O)$ denotes the semiclosure of O [3, 4, 5]. It was shown in [3] that α-sets and feebly open sets are the same sets in any spaces. Complement of a feebly open set is called a feebly closed set. Intersection of all feebly open sets containing S is called the feeble closure of S and is denoted by $f\mathrm{cl}(S)$.

By $RO(X,T)$, $DO(X,T)$ and $FO(X,T)$ ($RO(X)$, $DO(X)$ and $FO(X)$ without confusions), we will denote, respectively, the family of all regular open sets, all δ-open sets and all feebly open sets of a space (X,T).

2. Characterization

DEFINITION 2.1. A function $g: X \to Y$ is said to be na-continuous if, for each

$V \in FO(Y)$, $g^{-1}(V) \in DO(X)$.

THEOREM 2.1. *For a function $g: X \to Y$, the followings are equivalent:*

(a) *g is na-continuous.*

(b) *For each $x \in X$ and each $V \in FO(Y)$ containing $g(x)$, there exists an $U \in DO(X)$ containing x such that $g(U) \subset V$.*

(c) *For each $x \in X$ and each $V \in FO(Y)$ containing $g(x)$, there exists an $U \in RO(X)$ containing x such that $g(U) \subset V$.*

(d) *For each feebly closed set F of Y, $g^{-1}(F)$ is δ-closed.*

(e) *$g(\delta\mathrm{cl}(A)) \subset f\mathrm{cl}(g(A))$ for each subset A of X.*

(f) *$\delta\mathrm{cl}(g^{-1}(B)) \subset g^{-1}(f\mathrm{cl}(B))$ for each subset B of Y.*

PROOF. (a)$\Rightarrow$(b) : Let $x \in X$ and $V \in FO(Y)$ containing $g(x)$. Then $g^{-1}(V) \in FO(X)$ containing x. Put $U = g^{-1}(V)$. Then we have $g(U) \subset V$.

(b)$\Rightarrow$(c) : Let $x \in X$ and $V \in FO(Y)$ containing $g(x)$. Then there exists an $U_0 \in DO(X)$ containing x such that $g(U_0) \subset V$. Since a δ-open set is the union of regular open sets [12], there exists an $U \in RO(X)$ such that $x \in U \subset U_0$. Therefore, we have $g(U) \subset V$.

(c)$\Rightarrow$(d) : Let F be a feebly closed set of Y. For each $x \in g^{-1}(Y-F)$, there exists an $U_x \in RO(X)$ such that $x \in U_x \subset g^{-1}(Y-F)$. Therefore, we have $g^{-1}(F) = \cap\{X - U_x \mid x \in g^{-1}(Y-F)\}$. This means that $g^{-1}(F)$ is δ-closed in X.

(d)$\Rightarrow$(e) : For each subset A of X, $f\mathrm{cl}(g(A))$ is a smallest feebly closed set of Y containing $g(A)$ [4, Theorem 2.10]. Thus $A \subset g^{-1}(f\mathrm{cl}(g(A)))$ and hence $\delta\mathrm{cl}(A) \subset g^{-1}(f\mathrm{cl}(g(A)))$ by (d). Therefore, we obtain $g(\delta\mathrm{cl}(A)) \subset f\mathrm{cl}(g(A))$.

(e)$\Rightarrow$(f) : For each $B \subset Y$, we have $g(\delta\mathrm{cl}(g^{-1}(B))) \subset f\mathrm{cl}(g(g^{-1}(B))) \subset f\mathrm{cl}(B)$ and hence $\delta\mathrm{cl}(g^{-1}(B)) \subset g^{-1}(f\mathrm{cl}(B))$.

(f)$\Rightarrow$(a) : Let $V \in FO(Y)$. Then $Y-V$ is a feebly closed set and $\delta\mathrm{cl}(g^{-1}(Y-V)) \subset g^{-1}(f\mathrm{cl}(Y-V)) = g^{-1}(Y-V)$. Thus $g^{-1}(Y-V)$ is δ-closed in X and $g^{-1}(V) \in DO(X)$.

DEFINITION 2.2. A function $g: X \to Y$ is said to be super-continuous [10] if for each $x \in X$ and each neighborhood V of $g(x)$, there exists a neighborhood U of x such that $g(\mathrm{int}(\mathrm{cl}(U))) \subset V$.

It was shown in [3] that $FO(X)$ is a topology on a space X, that is, $(X, FO(X))$ is a topological space. $RO(X)$ is a basis for a topology which is called the semi-regularization of T for a space (X, T) and denoted by T_s.

THEOREM 2.2. *For a function $g : (X,T) \to (Y,T')$, the followings are equivalent:*

(a) *g is na-continuous.*

(b) *$g_0 : (X,T) \to (Y, FO(Y))$ is super-continuous, where $g_0(x)=g(x)$ for $x \in X$.*

(c) *$g_* : (X,T_s) \to (Y, FO(Y))$ is continuous, where $g_*(x)=g(x)$ for $x \in X$.*

PROOF. (a)$\Rightarrow$(b) : Let V be an open set of $(Y, FO(Y))$. Then $V \in FO(Y,T)$ and $g^{-1}(V) \in DO(X)$. It follows from Theorem 2.1 in [10] that g is super-continuous.

(b)$\Rightarrow$(c) : For each open set V of $(Y, FO(Y))$, $g_0^{-1}(V) \in DO(X,T)$ and $g_*^{-1}(V)$ is open in (X,T_s). Therefore, g_* is continuous.

(c)$\Rightarrow$(a) : For each $V \in FO(Y,T')$, V is open in $(Y, FO(Y))$ and hence $g_*^{-1}(V)$ is open (X,T_s). Therefore, $g^{-1}(V) \in DO(X,T)$. Hence g is na-continuous.

DEFINITION 2.3. A filterbase $\mathscr{B}=\{B_\lambda\}$ on a space X is said to δ-converge (resp. sf-converge [3]) to a point x in X [8] if, for each $V \in RO(X)$ (resp. $V \in FO(X)$) there exists a $B_\lambda \in \mathscr{B}$ such that $B_\lambda \subset V$.

A net $\{x_\lambda\}_{\lambda \in D}$ in X is said to δ-converge (resp. sf-converge) to $x \in X$ if the net is eventually in each regular open set containing x (resp. each feebly open set).

THEOREM 2.3. *For a function $g : (X,T) \to (Y,T')$, the followings are equivalent:*

(a) *g is na-continuous.*

(b) *For each $x \in X$ and each filterbase $\mathscr{B}$ δ-converging to x, $g(\mathscr{B})$ converges to $g(x)$ in $(Y, FO(Y))$.*

(c) *For each $x \in X$ and each net $\{x_\lambda\}_{\lambda \in D}$ δ-converging to x, the net $\{g(x_\lambda)\}_{\lambda \in D}$ converges to $g(x)$ in $(Y, FO(Y))$.*

(d) *For each $x \in X$ and each filterbase $\mathscr{B}$ δ-converging to x, $g(\mathscr{B})$ sf-converges to $g(x)$ in (Y,T').*

(e) *For each $x \in X$ and each net $\{x_\lambda\}_{\lambda \in D}$ δ-converging to x, the net $\{g(x_\lambda)\}_{\lambda \in D}$ sf-converges to $g(x)$ in (Y,T').*

PROOF. (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c) follows, immediately, from Theorem 2.1 and (a)$\Leftrightarrow$(d)$\Leftrightarrow$(e) follows, easily, from Theorem 2.1.

3. Basic properties

THEOREM 3.1. *If $g : X \to Y$ is na-continuous and A is open in X, then the restriction $g|A : A \to Y$ is na-continuous.*

PROOF. Let $V \in FO(Y)$. Then $g^{-1}(V) \in DO(X)$ and it is a union of regular open sets V_t of X. Since A is open in X, $V_t \cap A$ is regular open in the subspace A [12, Theorem 4]. Therefore, $(g|A)^{-1}(V)$ is the union of $g^{-1}(V_t) \cap A$ and hence $(g|A)^{-1}(V) \in DO(A)$.

THEOREM 3.2. *If $g : X \to Y$ and $f : Y \to Z$ are na-continuous, then the composition $f \circ g : X \to Z$ is na-continuous.*

PROOF. It is obvious to prove, since each δ-open set is feebly open in a space.

LEMMA 3.1. *Let $\{X_\lambda \mid \lambda \in D\}$ be a family of spaces and $U_{\lambda i}$ be a subset of $X_{\lambda i}$ for each $i = 1, 2, \cdots, n$. Then $U = \prod_{i=1}^{n} U_{\lambda i} \times \prod_{\lambda \neq \lambda i} X_\lambda$ is δ-open (resp. feebly open) in $\prod_{\lambda \in D} X_\lambda$ if and only if $U_{\lambda i} \in DO(X_{\lambda i})$ (resp. $U_{\lambda i} \in FO(x_{\lambda i})$) for each $i = 1, 2, \cdots, n$.*

PROOF. From [7, Remark 2.1], $(\prod X_\lambda)_s = \prod (X_\lambda)_s$. Thus $U \in DO(\prod X_\lambda)$ iff U is open in $(\prod X_\lambda)_s$ where $(X_\lambda)_s$ is the semi-regularizaion of X_λ. Therefore, $U \in DO(\prod X_\lambda)$ iff $U_{\lambda i}$ is open in $(X_{\lambda i})_s$ i.e., $U_{\lambda i} \in DO(X_{\lambda i})$ for each $i = 1, 2, \cdots, n$.

Next, assume that $U \in FO(\prod X_\lambda)$. Then, by [6, Problem 1—2, p. 105], we have $U \subset \mathrm{int}\,(\mathrm{cl}(\mathrm{int}(U))) \subset \{\prod_{i=1}^{n} \mathrm{int}(\mathrm{cl}(\mathrm{int}(U_{\lambda i})))\} \times \prod_{\lambda \neq \lambda i} X_\lambda$. Therefore, we obtain $U_{\lambda i} \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(U_{\lambda i})))$ for each $i = 1, 2, \cdots, n$. Thus $U_{\lambda i} \in FO(X_{\lambda i})$ for each $i = 1, 2, \cdots, n$. Conversely, assume that $U_{\lambda i} \in FO(X_{\lambda i})$ for each $i = 1, 2, \cdots, n$. Then, $U \subset \{\prod_{i=1}^{n} \mathrm{int}(\mathrm{cl}(\mathrm{int}(U_{\lambda i})))\} \times \prod_{\lambda \neq \lambda i} X_\lambda \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(U)))$. The proof completes.

THEOREM 3.3. *Let $g_\lambda : X_\lambda \to Y_\lambda$ be a function for each $\lambda \in D$ and $g : \prod X_\lambda \to \prod Y_\lambda$ a function defined by $g(\{x_\lambda\}) = \{g_\lambda(x_\lambda)\}$ for each $\{x_\lambda\} \in \prod X_\lambda$. If g is na-continuous, then g_λ is na-continuous for each $\lambda \in D$.*

PROOF. Let $\beta \in D$ and $V_\beta \in FO(Y_\beta)$. Then, by Lemma 3.1, $V = V_\beta \times \prod_{\lambda \neq \beta} Y_\lambda$ is feebly open in $\prod Y_\lambda$ and $g^{-1}(V) = g_\beta^{-1}(V_\beta) \times \prod_{\lambda \neq \beta} X_\lambda$ is δ-open in $\prod X_\lambda$. From Lemma 3.1, $g_\beta^{-1}(V_\beta) \in DO(X)$. Therefore, g_β is na-continuous.

THEOREM 3.4. *Let $g: X \to Y$ be a function and $G: X \to X \times Y$ be the graph function of g defined by $G(x)=(x, g(x))$ for each $x \in X$. If G is na-continuous, then g is na-continuous.*

PROOF. Let $x \in X$ and $V \in FO(Y)$ containing $g(x)$. Then, by Lemma 3.1, $X \times V \in FO(X \times Y)$ containing $g(x)$. Since G is na-continuous, by Theorem 2.1 there exists an $U \in DO(X)$ containing x such that $G(U) \subset X \times V$. Hence $g(U) \subset V$.

REMARK 3.1. It was known in [3, Example 2.2] that $V \in FO(X \times Y)$ may not, generally, be a union of sets of the form $A \times B$ in the product space $X \times Y$, where $A \in FO(X)$ and $B \in FO(Y)$. Therefore, the converse of Theorem 3.4 may not be true, generally.

4. Comparisions

In this section we investigate relations between na-continuity and several strong forms of continuity. We shall recall definitions of functions used here.

DEFINITION 4.1. A function $g: X \to Y$ is said to be strongly continuous (written STC) [9] if $g(\mathrm{cl}(A)) \subset g(A)$ for each subset A of X.

REMARK 4.1. It was shown in [9, Corollary 2] that $g: X \to Y$ is STC if and only if $g^{-1}(B)$ is open and closed in X for each $B \subset Y$.

DEFINITION 4.2. A function $g: X \to Y$ is said to be completely continuous (written CC) [1] (resp. β-continuous (written βC) [4]) if $g^{-1}(V) \in RO(X)$, for each open (resp. regular open) set V of Y.

D. Carnahan [2] called βC functions R-maps.

DEFINITION 4.3. A function $g: X \to Y$ is said to strongly θ-continuous (written STθ) [14] (resp. δ-continuous (written δC) [12], almost continuous (written AC) [15]) if, for each $x \in X$ and each open neighborhood V of $g(x)$, there exists an open neighborhood U of x such that $g(\mathrm{cl}(U)) \subset V$ (resp. $g(\mathrm{int}(\mathrm{cl}(U))) \subset \mathrm{int}(\mathrm{cl}(V))$, $g(U) \subset \mathrm{int}(\mathrm{cl}(V))$).

THEOREM 4.1. *The next implications hold but none of the implications may be, in general, reversible (Example 4.1 and 4.2.).*

Strongly Continuity ⇒ Na-Continuity ⇒ Super-Continuity ⇒ Continuity

PROOF. The lst implication follows from Remark 4.1. Let $g: X \to Y$ be na-continuous and V an open set of Y. Then $g^{-1}(V) \in DO(X)$ because open sets are feebly open. Hence g is super-continuous [10, Theorem 2.1.]. The 3rd implication was shown in [10, 14].

EXAMPLE 4.1. Let $X=\{a,b,c\}$ be a space with $T=\{\phi, \{a\}, \{b\}\ \{a,b\}, X\}$. Let $g: X \to X$ be a function defined by $g(a)=g(b)=a$ and $g(c)=c$. Then g is na-continuous because $T=FO(X)$. However, g is not STθ and hence not STC [13, Example 4.14]

EXAMPLE 4.2. Let R be the usual space of reals and $i: R \to R$ be the identity function. Since R is regular, i is STθ. However, since there exists a feebly open set which is not open in R [3, Example 2.1], i is not na-continuous. Therefore, a super-continuous function is, in general, not na-continuous.

REMARK 4.2. Na-continuity and β-continuity are independent of each other, as the next examples show.

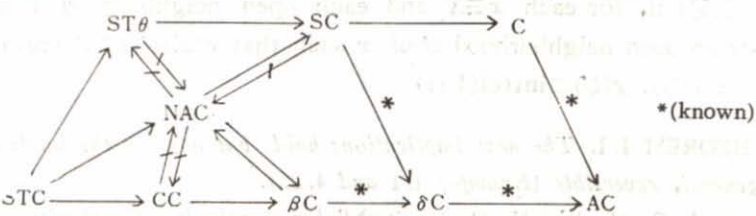
EXAMPLE 4.3. Let $X=\{a,b,c\}$ be a space with $T=\{\phi, \{a\}, X\}$. Then the identity $i: X \to X$ is βC, but not na-continuous.

EXAMPLE 4.4. Let R be the usual space of reals and $Y=\{a,b,c,d\}$ be a space with $T=\{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, Y\}$. Define a function $g: R \to Y$ by ; $g(x)=a$ if $x<p$; $g(x)=b$ if $p<x<q$; $g(x)=c$ if $q<x<r$; $g(x)=d$ if $x=p, q$ and $r \leqq x$, where p, q and r are distinct reals. Then g is na-continuous, but not βC and hence not CC.

REMARK 4.3. Complete continuity and na-continuity are independent, as Example 4.4 and the next example show.

EXAMPLE 4.5. Let $X=\{a,b,c,d\}$ be a space with $T=\{\phi, \{c\}, \{a,b\}, \{a,b,c\}, X\}$. Let $Y=\{x,y,z\}$ be a space with $T=\{\phi, \{x\}, Y\}$. Define a function $g: X \to Y$ by $g(a)=g(b)=x$, $g(c)=y$ and $g(d)=z$. Then g is CC, but not na-continuous.

From the results in this section, we obtain the following diagram:



where $A \not\to B$ represents that A does not always imply B, moreover, SC= super-continuous, NAC=na-continuous and C=continuous.

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