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ON THE SEPARATION AXIOM R_T

By Maria-Dolors Arnal and Josep Guia

1. Introduction

The axiom weaker than R_0 introduced by J. Tong (1983) has suggested us the definition of a family of new axioms in the way of the work of J. Guia (1984).

DEFINITION 1.1. In a topological space $(X, \mathscr{T})$, let x be a point of X. The *closure* of x is the set $\{\bar{x}\}=\bigcap\{F \mid x\in F,\ F \text{ closed}\}$; the *kernel* of x, $\{\hat{x}\}=\bigcap\{O \mid x\in O,\ O\in\mathscr{T}\}$; the *covering* of x, $\langle x\rangle=\{\bar{x}\}\cap\{\hat{x}\}$; the *derived* set of x, $\mathrm{d}\{x\}=\{\bar{x}\}-\{x\}$; the *essential derived* set of x, $\mathrm{D}\{x\}=\{\bar{x}\}-\langle x\rangle$; the *shell* of x, $\mathrm{s}\{x\}=\{\hat{x}\}-\{x\}$; the *essential shell* set of x, $\mathrm{S}\{x\}=\{\hat{x}\}-\langle x\rangle$.

DEFINITION 1.2. In a topological space $(X, \mathscr{T})$, a subset A of X is said to be essential degenerate if it is contained in $\langle x\rangle$ for some $x\in X$.

PROPOSITION 1.3. [1] *A topological space $(X, \mathscr{T})$ is a T_1-space iff one of the following conditions holds:*

1) $\forall x\in X,\ \{\bar{x}\}=\{x\}$
2) $\forall x\in X,\ d\{x\}=\phi$
3) $\forall x\in X,\ \{\hat{x}\}=\{x\}$
4) $\forall x\in X,\ s\{x\}=\phi$
5) $\forall x,y\in X,\ x\neq y$ *implies* $\{\bar{x}\}\cap\{\bar{y}\}=\phi$
6) $\forall x,y\in X,\ x\neq y$ *implies* $\{\hat{x}\}\cap\{\hat{y}\}=\phi$

PROPOSITION 1.4. *A topological space $(X, \mathscr{T})$ is a R_0-space iff one of the following conditions holds:*

1) $\forall x\in X,\ \{\bar{x}\}=\langle x\rangle$
2) $\forall x\in X,\ D\{x\}=\phi$

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3) $\forall x \in X$, $\{\hat{x}\} = \langle x \rangle$

4) $\forall x \in X$, $S\{x\} = \phi$

5) $\forall x, y \in X$, $\langle x \rangle \neq \langle y \rangle$ *implies* $\{\bar{x}\} \cap \{\bar{y}\} = \phi$

6) $\forall x, y \in X$, $\langle x \rangle \neq \langle y \rangle$ *implies* $\{\hat{x}\} \cap \langle \hat{y} \rangle = \phi$

PROOF.

It is immediate. See (1.1) and [2], [3], [4], [5].

DEFINITION 1.5. A topological space $(X, \mathscr{T})$ is a R'_T-space if, for every $x \in X$, D$\{x\}$ and S$\{x\}$ are degenerate.

REMARK 1.6. R'_T axiom has been introduced by J. Tong [10] as R_T axiom. We will see that our notation is better suited for the purpose since R'_T is stronger than the correspondent ET_T (see [6]) whereas R_α-axioms are weaker than ET_α-axioms (see [6] and [8])

PROPOSITION 1.7. [1] [7] *A topological space* $(X, \mathscr{T})$ *is a* T_{YS}*-space iff one of the following conditions holds*:

1) $\forall x, y \in X, x \neq y$ *implies* $\{\bar{x}\} \cap \{\bar{y}\}$ *is either* ϕ *or* $\{x\}$ *or* $\{y\}$.

2) $\forall x, y \in X, x \neq y$ *implies* $\langle x \rangle \neq \langle y \rangle$ *and* $d\{x\} \cap d\{y\} = \phi$.

DEFINITION 1.8. [6] A topological space $(X, \mathscr{T})$ is a R^*_{YS}-space if, for arbitrary $x, y \in X$, $x \neq y$ implies D$\{x\} \cap$ D$\{y\} = \phi$.

PROPOSITION 1.9. [8] [6] *A topological space* $(X, \mathscr{T})$ *is a* R'_{YS}*-space iff one of the following conditions holds*:

1) $\forall x, y \in X$, $\langle x \rangle \neq \langle y \rangle$ *implies* $\{\bar{x}\} \cap \{\bar{y}\}$ *is either* ϕ *or* $\{x\}$ *or* $\{y\}$.

2) $\forall x, y \in X$, $\langle x \rangle \neq \langle y \rangle$ *implies* $d\{x\} \cap d\{y\} = \phi$.

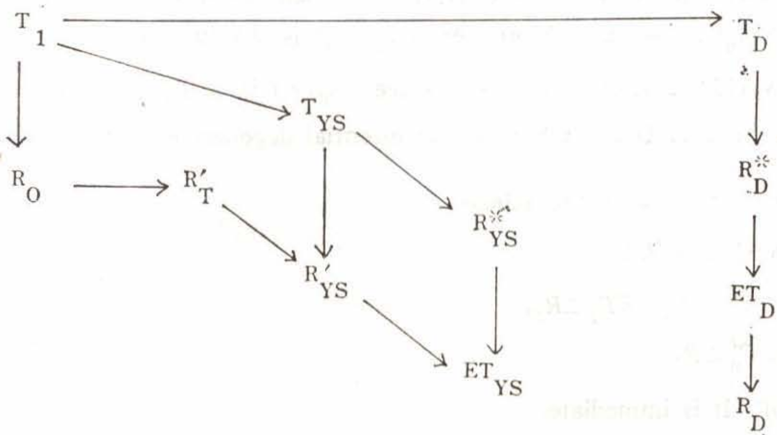
DEFINITION 1.10. A topological space $(X, \mathscr{T})$ is ET_α ("essentially T_α") if its T_0-identification space [9] is T_α.

PROPOSITION 1.11. [7] *A topological space* $(X, \mathscr{T})$ *is a* ET_{YS}*-space iff, for arbitrary* $x, y \in X$, $\langle x \rangle \neq \langle y \rangle$ *implies* $D\{x\} \cap D\{y\} = \phi$.

DEFINITION 1.12. [1] A topological space $(X, \mathscr{T})$ is a T_D-space if, for every $x \in X$, d$\{x\}$ is a closed set.

DEFINITION 1.13. [5] A topological space $(X, \mathscr{T})$ is a R^*_D-space if, for every $x \in X$, d$\{x\}$ not closed implies D$\{x\} = \phi$.

The following diagram shows the ordering between the mentioned axioms.



PROPOSITION 1.14. [5] *A topological space* $(X, \mathcal{T})$ *is a* ET_D*-space iff, for every* $x \in X$, $D\{x\}$ *is a closed set.*

DEFINITION 1.15. [8] A topological space $(X, \mathcal{T})$ is a R_D-space if, for every $x \in X$, $\langle x \rangle = \{x\}$ implies $d\{x\}$ is closed.

2. New axioms

DEFINITION 2.1. A topological space $(X, \mathcal{T})$ is a C_0^F-space if, for every $x \in X$, $D\{x\} \neq \phi$ or $S\{x\} \neq \phi$ implies $\langle x \rangle \neq \{x\}$.

DEFINITION 2.2. A topological space $(X, \mathcal{T})$ is a T_T-space if, for every $x \in X$, $d\{x\}$ and $s\{x\}$ are degenerate and $\langle x \rangle = \{x\}$.

PROPOSITION 2.3. *A topological space* $(X, \mathcal{T})$ *is a* ET_T*-space iff, for every* $x \in X$, $D\{x\}$ *and* $S\{x\}$ *are essential degenerate.*

PROOF. Let $(X_0, \mathcal{T}_0)$ be the T_0-identification space of $(X, \mathcal{T})$, let d_0 and s_0 be the correspondent derived and shell operators, and let π be the projection map from $(X, \mathcal{T})$ onto $(X_0, \mathcal{T}_0)$.

If $(X, \mathcal{T})$ is ET_T then $(X_0, \mathcal{T}_0)$ is T_T and, hence, $d_0\{\langle x \rangle\}$ and $s_0\{\langle x \rangle\}$ are degenerate for every $\langle x \rangle \in X_0$. If $D\{x\}$ is not essential degenerate then it is a union of distinct coverings. Since $d_0\{\langle x \rangle\} = \pi(D\{x\})$ (see [7]), $d_0\{\langle x \rangle\}$ is not degenerate. Similarly, if we assume that $S\{x\}$ is not essential degenerate, then $s_0\{\langle x\rangle\}$ is not degenerate.

Conversely, if $D\{x\}=\langle y\rangle$ and $S\{x\}=\langle z\rangle$ then $\pi(D\{x\})=d_0\{\langle x\rangle\}=\{\langle y\rangle\}$ and $\pi(S\{x\})=s_0\{\langle x\rangle\}=\{\langle z\rangle\}$. Moreover, $(X_0, \mathscr{T}_0)$ is obviously T_0.

DEFINITION 2.4. A topological space $(X, \mathscr{T})$ is a R_T-space if, for every $x\in X$, whenever $D\{x\}$ or $S\{x\}$ is not essential degenerate, then $\langle x\rangle\neq\{x\}$.

3. Relations between the axioms

PROPOSITION 3.1.

1) $T_1\subset T_T\subset R'_T\subset ET_T\subset R_T$,

2) $R_0\subset C_0^F\subset R_T$

PROOF. It is immediate.

LEMMA 3.2. *In a topological space* $(X, \mathscr{T})$, *for arbitrary* $x, y\in X$,

1) $x\in D\{y\}$ *iff* $y\in S\{x\}$

2) $x\in d\{y\}$ *iff* $y\in s\{x\}$

PROOF. It is immediate.

LEMMA 3.3. *In a topological space* $(X, \mathscr{T})$, *for every* $x\in X$,

1) $D\{x\}$ *essential degenerate implies* $D\{x\}$ *closed*

2) $d\{x\}$ *degenerate and* $\langle x\rangle=\{x\}$ *implies* $d\{x\}$ *closed.*

PROOF. Statement 1) follows from the fact that $D\{x\}$ is a union of closed sets and statement 2) follows from 1) through the canonical projection π.

PROPOSITION 3.4.

1) $T_T\subset T_D$, 2) $R'_T\subset R^*_D$, 3) $ET_T\subset ET_D$, 4) $R_T\subset R_D$.

PROOF. 1), 3) and 4) follow from lemma (3.3).

Let $(X, \mathscr{T})$ be a R'_T-space and let x be a point of X. Assume that $D\{x\}$ is degenerate and not empty, that is $D\{x\}=\{y\}$. If $d\{x\}$ is not closed then, from lemma 3.3, $d\{x\}$ is not degenerate and $\langle x\rangle\neq\{x\}$. Hence, there exists $z\in X$ such that $z\neq x$ and $z\in\langle x\rangle$. From lemma 3.2, $x\in S\{y\}$, and it is clear that $z\in S\{y\}$, in contradiction with the hypothesis.

PROPOSITION 3.5.

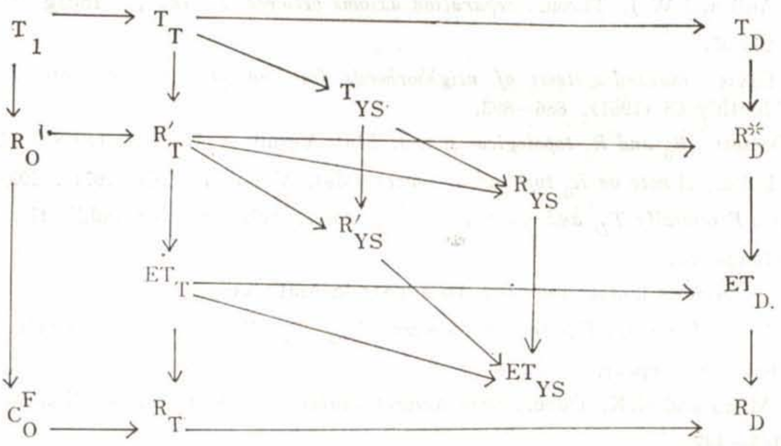
1) $T_T\subset T_{YS}$ 2) $R'_T\subset R^*_{YS}$ 3) $ET_T\subset ET_{YS}$

PROOF. Let $(X, \mathscr{T})$ be a T_T-space. If for two distinct points x, y belonging X, there exists $z \in \mathrm{d}\{x\} \cap \mathrm{d}\{y\}$ then, from lemma 3.2, $x, y \in \mathrm{s}\{z\}$, in contr adiction with the hypothesis.

Statement 2) may be proved in the same way.

Let $(X, \mathscr{T})$ be a ET_T-space. If for two points $x, y \in X$ such that $\langle x\rangle \neq \langle y\rangle$ there exists $z \in \mathrm{D}\{x\} \cap \mathrm{D}\{y\}$ then $\mathrm{D}\{x\} = \mathrm{D}\{y\} = \langle z\rangle$. Hence, from lemma 3.2, $x, y \in \mathrm{S}\{z\}$ and $\langle x\rangle$, $\langle y\rangle \in \mathrm{S}\{z\}$, in contradiction with the hypothesis.

The following diagram shows the ordering relation between all the above axioms.



All the axioms are distinct. In the following examples the space X is the set of real numbers and it is understood that the null set and the set X are closed.

EXAMPLE 3.6. Let the closed sets be $\{x\}$, $\{-x, x\}$ $\{x \geq 0\}$ and their finite unions. This space is T_T but not C_0^F.

EXAMPLE 3.7. Let the closed sets be $\{x, 0\}$, $\{x, -x, 0\}$ $(x \geq 0)$ and their finite unions. This space is T_D but not ET_{YS} or R_T.

EXAMPLE 3.8. Let the closed sets be $\{x\}$ $(x \neq 0)$ and their finite unions. This space is T_{YS} but not R_T.

EXAMPLA 3.9. Let the closed sets be $\{-x, x\}$ $(x \geq 0)$ and their finite unions. This space is R'_T but not T_T.

EXAMPLE 3.10. Let the closed set be $\{0\}$. This space is ET_T but not R'_T.

EXAMPLE 3.11. Let the closed sets be $\{0, 1, x\}$ $(x \in R)$ and their finite unions. This space is R_T but not ET_T.

Facultat de Matemàtiques
Universitat de València
Burjassot (València), Spain.

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