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θ-IRREDUCIBLE SPACES

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A topological space is called *irreducible* [1] if every pair of nonempty open subsets of the space has a nonempty intersection. Irreducible spaces are also known under the name of *D-spaces* [4], *hyperconnected* spaces [6] and *S-connected* spaces [7]. The purpose of this note is to introduce the class of θ-irreducible spaces containing the class of irreducible spaces and to investigate its properties.

Throughout, X and Y will denote spaces and cl A (resp. int A) will denote the closure (resp. interior) of a subset A of a space. A subset A of a space X is said to be *regular-closed* (resp. *regular-open*) if cl int $A=A$ (resp. int cl $A=A$). The topology on X which has as its base the family of regular-open subsets of a space X is called the *semiregularization* topology. A topological property is called *semiregular* if it is shared by a space and its semiregularization. A point $x\in X$ is in the *θ-closure* [8] of a subset A of a space $X(x\in \mathrm{cl}_\theta A)$ if $\mathrm{cl}\, U\cap A\neq\phi$ for each open set U containing x.

DEFINITION 1. A space X is *θ-irreducible* if every pair of nonempty regular-closed subsets of X has a nonempty intersection.

It is clear that irreducible spaces are θ-irreducible. Example 75 of [6] shows that there are θ-irreducible spaces which are Hausdorff and hence are not irreducible. We also point out that θ-irreducible spaces are not Urysohn.

Our first result is a restatement of the definition of θ-irreducibility.

THEOREM 1. *A space X is θ-irreducible if and only if* $\mathrm{cl}_\theta \mathrm{cl}\, U=X$ *for each nonempty open subset U of X.*

Recall that a space X is *almost-regular* [5] if for each $x\in X$ and each regular-closed subset F of X such that $x\notin F$ there exists disjoint open sets U and V such that $x\in U$ and $F\subset V$.

THEOREM 2. *An almost-regular space is irreducible if and only if it is θ-irreducible.*

It is known that a space X is irreducible if and only if it is connected and extremally disconnected. We define a space X to be *extremally θ-disconnected* if $\mathrm{cl}_\theta \mathrm{cl}\, U$ is open for each open subset U of X and have the analogous characterization of θ-irreducibility.

THEOREM 3. *A space X is θ-irreducible if and only if X is connected and extremally θ-disconnected.*

THEOREM 4. *θ-irreducibility is semiregular.*

PROOF. This follows from the fact that a space and its semiregularization have the same regular-closed subsets.

The following lemma is easily established.

LEMMA 1. *Let A be a subset of a space X and B be a subset of A. Then* $\mathrm{cl}_\theta^A B \subset A \cap \mathrm{cl}_\theta B$.

A topological property P is called *contagious* [2] if a space X has P when a dense subset of X has P.

THEOREM 5. *θ-irreducibility is contagious.*

PROOF. Let D be a dense θ-irreducible subspace of a space X. Let V be a nonempty open subset of X. Then $V \cap D \neq \phi$ and since $V \cap D$ is open in D, by Theorem 1 it follows that $\mathrm{cl}_\theta^D \mathrm{cl}^D (V \cap D) = D$. Therefore $D = \mathrm{cl}_\theta^D (D \cap \mathrm{cl}(V \cap D))$ and consequently, $D = \mathrm{cl}_\theta^D (D \cap \mathrm{cl} V)$. By Lemma 1 it follows that $D \subset D \cap \mathrm{cl}_\theta (D \cap \mathrm{cl} V)$. Therefore $D \subset \mathrm{cl}_\theta \mathrm{cl} V$. This gives that $\mathrm{cl}_\theta \mathrm{cl} V = X$ since $\mathrm{cl}_\theta \mathrm{cl} V$ is closed and D is dense in X. By Theorem 1, X is θ-irreducible.

A function $f: X \to Y$ is θ-continuous [3] if for each $x \in X$ and each open set V containing $f(x)$ there exists an open set U containing x such that $f(\mathrm{cl} U) \subset \mathrm{cl} V$. Since the identity function from a space into its semiregularization as well as its inverse are both θ-continuous, the following result generalizes Theorem 4.

THEOREM 6. *θ-irreducibility is preserved under θ-continuous surjections.*

PROOF. Let $f: X \to Y$ be a θ-continuous surjection, let X be θ-irreducible, and let V be a nonempty open subset of Y. Since f is surjective there is an $x \in X$ such that $f(x) \in V$. The θ-continuity of f implies that there exists an open set U containing x such that $f(\mathrm{cl} U) \subset \mathrm{cl} V$. Since it is known that a function $f: X \to Y$ is θ-continuous if and only if $f(\mathrm{cl}_\theta A) \subset \mathrm{cl}_\theta f(A)$ for each subset

A of X, $f(\mathrm{cl}_\theta\mathrm{cl}\ U)\subset\mathrm{cl}_\theta\ f(\mathrm{cl}U)$. Therefore $f(\mathrm{cl}_\theta\mathrm{cl}U)\subset\mathrm{cl}_\theta\mathrm{cl}V$ and since X is θ-irreducible, by Theorem 1 it follows that $Y=f(X)=f(\mathrm{cl}_\theta\mathrm{cl}U)\subset\mathrm{cl}_\theta\mathrm{cl}V$. This shows that Y is θ-irreducible.

LEMMA 2. *Let $\{X_i : i\in I\}$ be family of spaces and let $A_i\subset X$ for each $i\in I$. Then $cl_\theta\prod\{A_i : i\in I\}=\prod\{cl_\theta A_i : i\in I\}$.*

PROOF. Let $(x_i)\in\mathrm{cl}_\theta\prod A_i$ and let $x_i\in U_i$, where U_i is open in X_i. Since $(x_i)\in U_i\times\prod_{j\neq i}X_j$ which is open in $\prod X_i$,

$$\phi\neq\mathrm{cl}(U_i\times\prod_{j\neq i}X_j)\cap\prod A_i=(\mathrm{cl}\ U_i\times\prod_{j\neq i}X_j)\cap\prod A_i=(\mathrm{cl}\ U_i\cap A_i)\times\prod_{j\neq i}A_j.$$

Therefore $\mathrm{cl}\ U_i\cap A_i\neq\phi$ and $x_i\in\mathrm{cl}_\theta A_i$. This shows that $(x_i)\in\prod\mathrm{cl}_\theta A_i$ and hence $\mathrm{cl}_\theta\prod A_i\subset\prod\mathrm{cl}_\theta A_i$. To establish the converse inclusion, let $(x_i)\in\prod\mathrm{cl}_\theta A_i$. Then for each basic open set $U_{i_1}\times\cdots\times U_{i_n}\times\prod_{i\neq i_n}X_i$ containing (x_i), each $\mathrm{cl}\ U_{i_j}\cap A_{i_j}\neq\phi$ so that

$$\mathrm{cl}(U_{i_1}\times\cdots\times U_{i_n}\times\prod_{i\neq i_n}X_i)\cap\prod A_i=(\mathrm{cl}U_{i_1}\times\cdots\times\mathrm{cl}U_{i_n}\times\prod_{i\neq i_n}X_i)\cap\prod A_i$$
$$=(\mathrm{cl}U_{i_1}\cap A_{i_1})\times\cdots\times(\mathrm{cl}U_{i_n}\cap A_{i_n})\times\prod_{i\neq i_n}A_i\neq\phi.$$

Therefore $(x_i)\in\mathrm{cl}_\theta\prod A_i$ and $\prod\mathrm{cl}_\theta A_i\subset\mathrm{cl}_\theta\prod A_i$.

THEOREM 7. *θ-irreducibility is productive.*

PROOF. Let $\{X_i : i\in I\}$ be a family of spaces. Suppose that $\prod X_i$ is θ-irreducible. Since the projections are continuous, by Theorem 6 it follows that each X_i is θ-irreducible.

Conversely, assume that each X_i is θ-irreducible and let U be a nonempty open basic set in X_i. Then $U=U_{i_1}\times\cdots\times U_{i_n}\times\prod_{i\neq i_n}X_i$, and since $\mathrm{cl}U=\mathrm{cl}U_{i_1}\times\cdots\times\mathrm{cl}U_{i_n}\times\prod_{i\neq i_n}X_i$, by Lemma 2 it follows that $\mathrm{cl}_\theta\mathrm{cl}U=\mathrm{cl}_\theta\mathrm{cl}U_{i_1}\times\cdots\times\mathrm{cl}_\theta\mathrm{cl}U_{i_n}\times\prod_{i\neq i_n}X_i$. Since X_i is θ-irreducible, $\mathrm{cl}_\theta\mathrm{cl}U_{i_j}=X_{i_j}$ by Theorem 1 and hence $\mathrm{cl}_{\theta\mathrm{c}}\mathrm{l}U=\prod X_i$. Therefore $\prod X_i$ is θ-irreducible.

It is known that real valued continuous functions on irreducible spaces are constant. Our final result shows that this fact is also true in the case of θ-irreducible spaces. Recall that a function $f: X\to Y$ has a θ-closed graph if the graph of f is a θ-closed subset of the product space $X\times Y$.

THEOREM 8. *The following are equivalent for a space X.*

(a) *X is θ-irreducible.*

(b) *X is extremally θ-disconnected and for every space Y every θ-continuous* *function $f: X\to Y$ with a θ-closed graph is constant.*

(c) *X is extremally θ-disconnected and for every Urysohn space Y every θ-continuous function $f: X\to Y$ is constant.*

(d) *X is extremally θ-disconnected and every real valued continuous function on X is constant.*

PROOF. Let X be a θ-irreducible and let $f: X\to Y$ be a θ-continuous function with a θ-closed graph. Suppose that there exist $x, y\in X$ such that $f(x)\neq f(y)$. Since f has a θ-closed graph, there exist an open set U containing x and an open set V containing $f(y)$ such that $\mathrm{cl}U\cap f^{-1}(\mathrm{cl}V)=\phi$. The θ-continuity of f implies that there is an open set W containing y such that $\mathrm{cl}W\subset f^{-1}(\mathrm{cl}V)$. Therefore $\mathrm{cl}U\cap\mathrm{cl}W=\phi$ which contradicts the assumption that X is θ-irreducible. So f is constant. Since by Theorem 3, X is extremally θ-disconnected, (a) implies (b). From the fact that θ-continuous functions into Urysohn spaces have θ-closed graphs it follows that (b) implies (c). It is clear that (c) implies (d). To show that (d) implies (a), suppose that X is not θ-irreducible. Then there exists a nonempty set U of X such that $V=\mathrm{cl}_\theta\mathrm{cl}U\neq X$. Since X is extremally θ-disconnected, V is open. Therefore V is both open and closed. Let $f: X\to R$ be a function defined by $f(x)=0$ if $x\in V$ and $f(x)=1$ if $x\in X-V$. Clearly f is continuous. This contradicts the assumption that every real valued continuous function on X is constant. Therefore X is θ-irreducible.

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